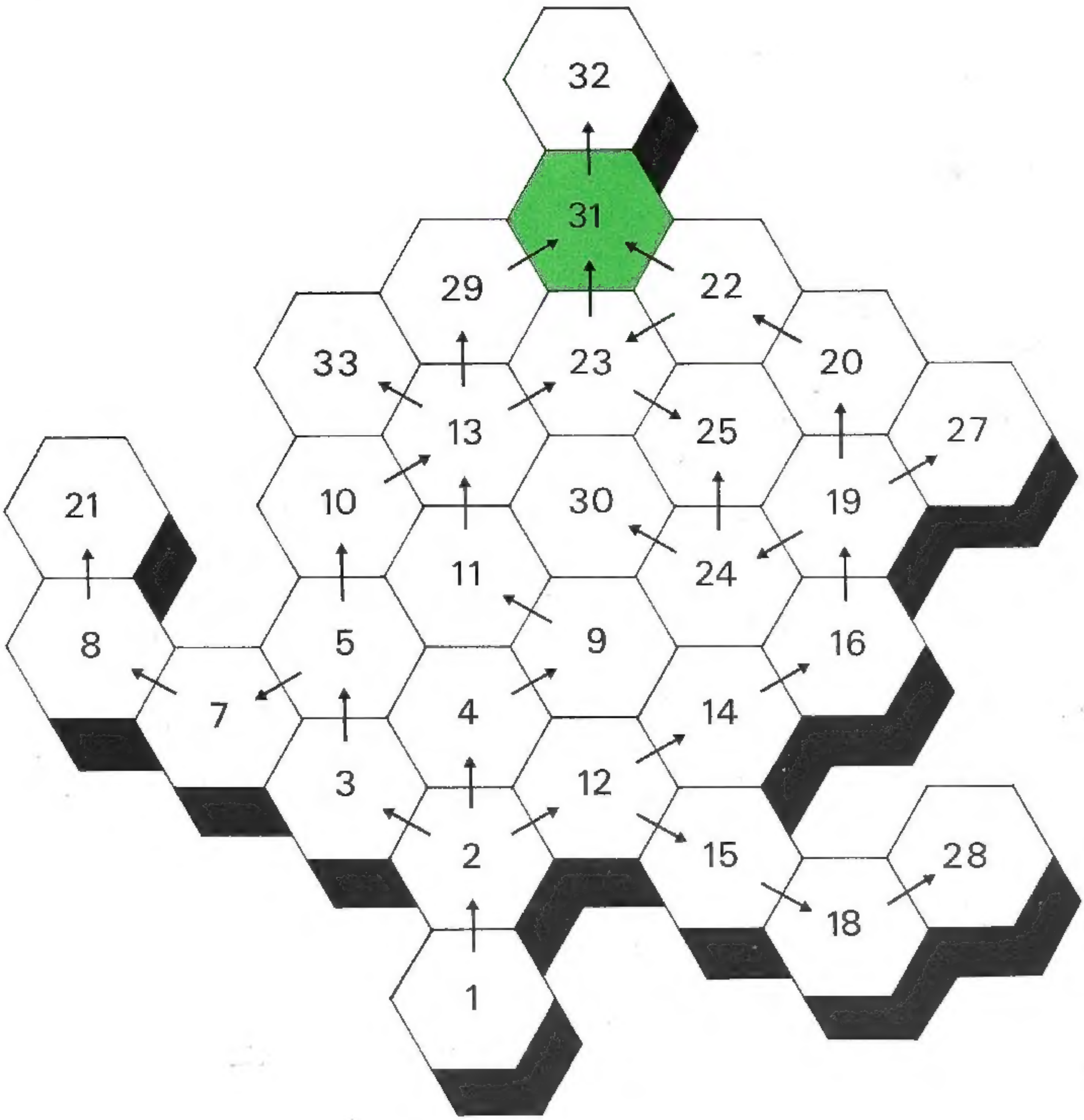
M201 31

THE OPEN UNIVERSITY

Mathematics: A Second Level Course

Linear Mathematics Unit 31

Fourier Transforms



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Mathematics: A Second Level Course

Linear Mathematics Unit 31

FOURIER TRANSFORMS

Prepared by the Linear Mathematics Course Team

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Further information on Open University courses may be obtained from the Admissions Office, The Open University, P.O. Box 48, Walton Hall, Milton Keynes, MK7 6AB.

1.2

Contents

Set Books

D. L. Kreider, R. G. Kuller, D. R. Ostberg and F. W. Perkins, *An Introduction to Linear Analysis* (Addison-Wesley, 1966).

E. D. Nering, *Linear Algebra and Matrix Theory* (John Wiley, 1970).

It is essential to have these books; the course is based on them and will not make sense without them.

Conventions

Before working through this correspondence text make sure you have read *A Guide to the Linear Mathematics Course*. Of the typographical conventions given in the Guide the following are the most important.

The set books are referred to as:

K for *An Introduction to Linear Analysis*
N for *Linear Algebra and Matrix Theory*

All starred items in the summaries are examinable.

References to the Open University Mathematics Foundation Course Units (The Open University Press, 1971) take the form *Unit M100 3, Operations and Morphisms*.

Note

Please note that this text is not based on the set books for the course.

31.0 INTRODUCTION

In *Unit 23, The Wave Equation*, we used two distinct methods to solve the wave equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{a^2}\frac{\partial^2 u}{\partial t^2}$$

The first method, that of separation of variables, is appropriate when the variable x is restricted to a finite interval, say $[0, \pi]$, and there are boundary conditions on $u(x, t)$ when x is at either end of this interval. (Let us note that independent of interval and boundary conditions, it is the *form* of an equation which allows us to separate it into ordinary differential equations for each variable separately. But it is the interval and the boundary conditions which determine whether we can find a useful solution thereby. Separation of variables gives the solution in the form of a series of functions.)

The second method, D'Alembert's solution, is appropriate when $x \in R$ and gives the solution in the form

$$u(x, t) = f(x + at) + g(x - at),$$

where f and g are functions of one variable.

When we try to solve other partial differential equations, for example the equation for heat conduction, which is considered in *Unit 32, The Heat Conduction Equation*, we find that the separation of variables method is still useful. However, D'Alembert's solution is not, being too closely dependent on the special characteristics of the wave equation. We have already seen a method which can be used in some of these cases, *viz.*, the Laplace transform. In this unit we shall discuss another transform method, one which is particularly suited to situations involving waves and other phenomena concerned with energy transfer. This is the method of *Fourier transforms*. It has other applications besides partial differential equations, for example in probability theory, quantum mechanics, and the theory of infinite groups, but the only application we consider in this course is to partial differential equations.

The basis of the method is to think of the infinite interval $(-\infty, \infty)$ as a limiting case of a finite interval $[-p, p]$. Before trying to solve any partial differential equations at all, we consider the simpler problem of applying this idea to an ordinary Fourier series. As we shall see, in the limit for large p, the Fourier series becomes an integral.

31.1 THE FOURIER INVERSION FORMULAS

31.1.1 The Fourier Cosine Transform

Let us carry out the limiting process outlined in the Introduction, for the Fourier series of a function f which satisfies certain restrictions imposed to simplify the calculation. (Some of them will be removed later, after we have established the basic formula.) The restrictions are

(i) f has domain and codomain R
(ii) f is continuous and has a derived function f' which is piecewise continuous (page K330). Further, f' has a derived function f'' which is piecewise continuous. That is, f' is piecewise smooth.
(iii) f is even: $f(-x) = f(+x)$ for all $x \in R$.
(iv) There is a positive real number p_0 such that $f(x) = 0$ for all $|x| > p_0$; when a function has this property we say that it is *finitary**.

From *Unit 22, Fourier Series* (Section 9-6 of K) we know how to represent such a function f as a Fourier series in any finite interval $[-p, p]$, where p is any positive number. The formula is

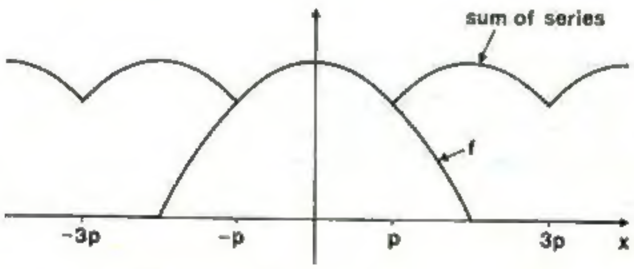
$$f(x) = \frac{1}{2}a_0 + a_1 \cos\frac{\pi x}{p} + a_2 \cos\frac{2\pi x}{p} + \cdots \qquad (x \in [-p, p]), \quad (1)$$

where

$$a_k = \frac{1}{p}\int_{-p}^{p} f(x)\cos\frac{k\pi x}{p}\,dx$$

$$= \frac{2}{p}\int_{0}^{p} f(x)\cos\frac{k\pi x}{p}\,dx \qquad (2)$$

The series (1) contains no sine terms, because f is even, and converges *pointwise* to $f(x)$ for $x \in [-p, p]$ because of condition (ii). For x outside $[-p, p]$, however, the series may not converge to $f(x)$; indeed, the sum of the series defines a periodic function of x with period $2p$, whereas in general f is not periodic.



Clearly, the larger the value of p, the larger the interval over which we get a Fourier approximation. To get a representation that works for *all* x, we take the limit for large p of Formula (1). Then, whatever x is chosen, p will eventually be larger than $|x|$ and so x will lie inside $[-p, p]$; thus the formula holds, in the limit, for that value of x. This argument gives

$$f(x) = \lim_{p\to\infty}\left(\frac{1}{2}a_0 + a_1\cos\frac{\pi x}{p} + a_2\cos\frac{2\pi x}{p} + \cdots\right) \qquad (x \in R) \quad (3)$$

where "$p \to \infty$" means the same as "p large." (The justification of Equation 3 is considered in Exercise 3 of this sub-section.)

To calculate the limit (3) we need to know how the a_k depend on p for large p. This can be found from Equation (2), using the fact that f is finitary. For then there is a number p_0 such that

$$f(x) = 0 \text{ for } |x| > p_0.$$

* Some authors use the word *finitary* to cover (i), (ii) and (iv).

Then if $p > p_0$, we have $f(x) = 0$ for $|x| > p$, so that in this case, Equation (2) reduces to

$$a_k = \frac{2}{p}\int_0^\infty f(x)\cos\frac{k\pi x}{p}\,dx.$$

Thus we may write

$$a_k = \frac{2}{p}F_c\left(\frac{k\pi}{p}\right) \quad \text{if} \quad p > p_0 \tag{4}$$

where

$$F_c(y) = \int_0^\infty f(x)\cos(yx)\,dx \qquad (y \in R) \tag{5}$$

This is an important definition. We call the function F_c the *Fourier cosine transform* of f.

$$F_c : y \longmapsto \int_0^\infty f(x)\cos(yx)\,dx \qquad (y \in R)$$

Using the formula (4) in (3) we can write the limit we want to calculate in the form

$$f(x) = \lim_{p\to\infty}\frac{2}{p}\left(\frac{1}{2}F_c(0) + F_c\left(\frac{\pi}{p}\right)\cos\frac{\pi x}{p} + F_c\left(\frac{2\pi}{p}\right)\cos\frac{2\pi x}{p} + \cdots\right) \tag{6}$$

When p becomes large, the spacing between the successive points $\frac{\pi}{p}$, $\frac{2\pi}{p}$, ... in the domain of the function F_c becomes very small; these points cover the positive real line more and more densely. This suggests replacing the sum in (6) by an integral. In *Unit M100 9, Integration I*, we defined the definite integral as follows

$$\int_a^b g = \lim_{n\to\infty} h\big(g(a) + g(a+h) + \cdots + g(a+(n-1)h)\big)$$

where $h = \frac{b-a}{n}$. It is reasonable to suppose that, for suitably well-behaved functions g this formula also holds in the limit for large b, i.e. that

$$\int_a^\infty g = \lim_{h\to 0} h\big(g(a) + g(a+h) + g(a+2h) + \cdots\big) \tag{7}$$

This is proved in Appendix 1 for functions satisfying conditions (i) to (iv). We can use (7) to simplify (6); we set

$$a = 0$$

$$h = \frac{\pi}{p}$$

and

$$g(kh) = F_c\left(\frac{k\pi}{p}\right)\cos\left(\frac{k\pi x}{p}\right),$$

i.e.

$$g(y) = F_c(y)\cos(xy).$$

Then (6) becomes

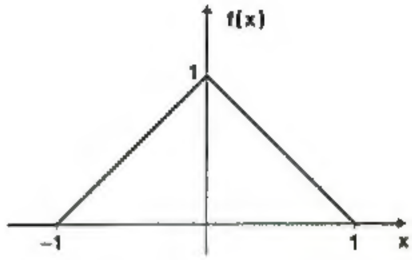
$$\begin{aligned} f(x) &= \lim_{h\to 0}\frac{2h}{\pi}\left(\tfrac{1}{2}F_c(0) + F_c(h)\cos(hx) + F_c(2h)\cos(2hx) + \cdots\right) \\ &= \frac{2}{\pi}\lim_{h\to 0} h\left(\tfrac{1}{2}g(0) + g(h) + g(2h) + \cdots\right) \\ &= \frac{2}{\pi}\lim_{h\to 0} h\big(g(0) + g(h) + \cdots\big) - \frac{2}{\pi}\lim_{h\to 0}\frac{h}{2}g(0) \\ &= \frac{2}{\pi}\int_0^\infty g + 0. \end{aligned}$$

Thus the limiting form of the Fourier cosine series for f is

$$f(x) = \frac{2}{\pi}\int_0^\infty F_c(y)\cos(xy)\, dy \tag{8}$$

A formula to recover a function from its transform is known as an *inversion formula*. In the case of Fourier cosine transforms, Equation (8) is the inversion formula. It tells us that the Fourier cosine transform of the Fourier cosine transform of f is just $\frac{\pi}{2}$ times f itself. Equation (5) shows us how to calculate the Fourier cosine transform of a given f; Equation (8) shows us how to recover f if we are given its Fourier cosine transform.

Example



The function f defined by

$$f(x) = \begin{Bmatrix} 1-|x| & \text{if} & |x|<1 \\ 0 & \text{if} & |x|\geqslant 1 \end{Bmatrix} \qquad (x \in R)$$

satisfies conditions (i) to (iv) (Check this.)
Its Fourier cosine transform is F_c, where

$$F_c(y) = \int_0^1 (1-x)\cos(xy)\, dx$$

$$= \frac{1-\cos y}{y^2} \qquad \text{(integration by parts).}$$

The inversion formula then gives us the complicated integral

$$\frac{2}{\pi}\int_0^\infty \frac{1-\cos y}{y^2}\cos(xy)\, dy = f(x) = \begin{cases} 1-|x| & \text{if } |x|<1 \\ 0 & \text{if } |x|\geqslant 1 \end{cases}$$

Exercises

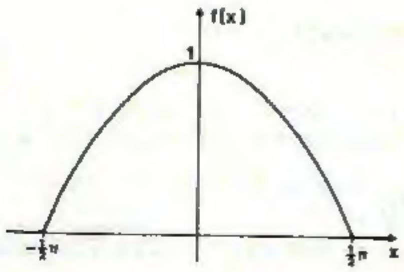
1. Follow through an argument analogous to the one given in the text above to obtain a formula giving f in terms of its *Fourier sine transform* F_s when f is *odd*: $f(-x) = -f(x)$. F_s is defined by

$$F_s(y) = \int_0^\infty f(x)\sin(xy)\, dx \qquad (y \in R).$$

2. Find the Fourier cosine transform of the function f defined by

$$f(x) = \begin{Bmatrix} \cos x & \text{if} & |x| < \frac{\pi}{2} \\ \\ 0 & \text{if} & |x| \geqslant \frac{\pi}{2} \end{Bmatrix} \qquad (x \in R)$$

and write down the *integral* obtained from the Fourier inversion formula.



3. Using the definition of a limit, prove that Equation (3) follows from Equation (1).
(*Hint:* consider an arbitrary fixed value of x, and take the limit for large p.)

Solutions

1. The changes necessary are:
Condition (iii) is replaced by

(iii)′ f is odd

whence (1) becomes

$$f(x) = b_1 \sin\frac{\pi x}{p} + b_2 \sin\frac{2\pi x}{p} + \cdots \qquad (x \in [-p, p])$$

and (3) becomes

$$f(x) = \lim_{p\to\infty}\left(b_1 \sin\frac{\pi x}{p} + b_2 \sin\frac{2\pi x}{p} + \cdots\right) \qquad (x \in R)$$

In the rest of the argument replace

a_0 by 0; a_k by b_k if $k \neq 0$; cos by sin; and F_c by F_s,

throughout. The final result is

$$f(x) = \frac{2}{\pi}\int_0^\infty F_s(y)\sin(xy)\,dy.$$

2.
$$\begin{aligned}
F_c(y) &= \int_0^{\pi/2} \cos x \cos(xy)\,dx \\
&= \int_0^{\pi/2} [\tfrac{1}{2}\cos(1+y)x + \tfrac{1}{2}\cos(1-y)x]\,dx \\
&\qquad\qquad\qquad\qquad \text{(Section III.5.1 of TI)} \\
&= \left[\frac{\sin(1+y)x}{2(1+y)} + \frac{\sin(1-y)x}{2(1-y)}\right]_0^{\pi/2} \qquad \text{if } y \neq \pm 1 \\
&= \frac{\sin\frac{1}{2}\pi(1+y)}{2(1+y)} + \frac{\sin\frac{1}{2}\pi(1-y)}{2(1-y)} \\
&= \frac{\cos\frac{1}{2}\pi y}{2(1+y)} + \frac{\cos(-\frac{1}{2}\pi y)}{2(1-y)} \\
&= \frac{\cos\frac{1}{2}\pi y}{1-y^2}.
\end{aligned}$$

For $y = \pm 1$, a separate calculation gives

$$F_c(\pm 1) = \frac{\pi}{4}.$$

The inversion formula gives

$$\frac{2}{\pi}\int_0^\infty \frac{\cos(\frac{1}{2}\pi y)\cos(xy)}{1-y^2}\,dy = \begin{cases} \cos x & \text{if } |x| < \dfrac{\pi}{2} \\ 0 & \text{if } |x| \geqslant \dfrac{\pi}{2} \end{cases}$$

3. Equation (1) tells us that

$$f(x) = C(p, x), \quad \text{if} \quad x \in [-p, p] \tag{9}$$

where

$$C(p, x) = \frac{1}{2}a_0 + \sum_{k=1}^{\infty} a_k \cos\left(\frac{k\pi x}{p}\right).$$

We wish to show that

$$f(x) = \lim_{p \to \infty} C(p, x) \qquad (x \in R)$$

i.e., for any $x \in R$ and for any positive number ε there exists a real number N such that

$$|f(x) - C(p, x)| < \varepsilon \text{ for all } p > N.$$

We can achieve this by choosing N to be $|x|$, for Equation (9) tells us that

$$f(x) - C(p, x) = 0 \text{ for all } p > |x|,$$

and 0 is certainly less than ε. Since this argument works for any real x, we have proved (3).

31.1.2 The Connection with Laplace Transforms

In the preceding sub-section we defined, for suitable types of function f, the Fourier cosine and sine transforms F_c and F_s:

$$F_c(y) = \int_0^\infty f(x)\cos(xy)\, dx \qquad (y \in R) \tag{1}$$

and

$$F_s(y) = \int_0^\infty f(x)\sin(xy)\, dx \qquad (y \in R) \tag{2}$$

Although we stated that f must be even in the first case and odd in the second, the actual integrals only involve $f(x)$ for $x \geqslant 0$, and so we can use these two expressions to define Fourier sine and cosine transforms even if f only has domain $[0, \infty)$ instead of $(-\infty, \infty)$—provided, of course, that the integrals converge. Of course, if f has domain $[0, \infty)$, it cannot be said to be either even or odd.

There is a similarity between the above two formulas and the formula defining the Laplace transform of f (*Unit 29, Laplace Transforms*):

$$\mathfrak{L}[f](y) = \int_0^\infty f(x)e^{-xy}\, dx \tag{3}$$

In all three cases $f(x)$ is multiplied by a particular expression involving y and x and the product is integrated. Since in each particular case the result depends on y only, it defines a new function (of y); such a new function is called an *integral transform* of f. Integral transforms provide a useful technique for solving various types of mathematical problem, as we have already seen in *Unit 29*. There are many other types of integral transform, for example the Mellin transform defined by

$$\int_0^\infty f(x)\, x^{y-1}\, dx;$$

or the Stieltjes transform

$$\int_0^\infty \frac{f(x)}{x+y}\,dx,$$

but we shall be concerned only with Fourier and Laplace transforms in this course. In all cases, we are dealing with a linear transformation on some relevant vector space of functions. The utility of these integral transforms depends upon the form of the differential equation to be solved.

Exercise

If K is a continuous function from $R \times R$ to R show that

$$A: f \longmapsto \int_a^b K(x, y) f(y)\,dy \qquad (f \in V)$$

is a linear transformation on V, the relevant real vector space of functions. (Note that K might be a Green's Function, Laplace transform, Stieltjes transform, or almost anything.)

Solution

We show that

$$A(\alpha f + \beta g) = \alpha A(f) + \beta A(g).$$

$$\begin{aligned} A(\alpha f + \beta g) &= \int_a^b K(x, y)(\alpha f + \beta g)(y)\,dy \\ &= \int_a^b K(x, y)[\alpha f(y) + \beta g(y)]\,dy \\ &= \int_a^b K(x, y)\alpha f(y)\,dy + \int_a^b K(x, y)\beta g(y)\,dy \\ &= \alpha \int_a^b K(x, y) f(y)\,dy + \beta \int_a^b K(x, y) g(y)\,dy \\ &= \alpha A(f) + \beta A(g). \end{aligned}$$

As you may have suspected from the similarity of the formulas defining them, a connection between the Fourier cosine and sine transforms on the one hand and the Laplace transform on the other may be found. This is because the sine, cosine and exponential functions are themselves connected by Euler's formula, which you met in *Unit M100 29, Complex Numbers II*:

$$\cos\theta + i\sin\theta = e^{i\theta} \tag{4}$$

This formula suggests that we consider the complex linear combination

$$\begin{aligned} F_c(y) + iF_s(y) &= \int_0^\infty f(x)\,[\cos(xy) + i\sin(xy)]\,dx \\ &= \int_0^\infty f(x)e^{ixy}\,dx \\ &= \mathfrak{L}[f](-iy) \end{aligned} \tag{5}$$

This formula provides the connection between Fourier and Laplace transforms. There are several points to be careful about here.

(i) The formula (5) only works if f has domain $[0, \infty)$ and all the integrals converge.

(ii) The integrand is now a special sort of *complex* function; its codomain is C, the set of complex numbers, and its domain is R. We

define the integral of such a complex function by treating the real and imaginary parts separately: if $g = h + ik$, where h and k both have R as domain and codomain, then

$$\int_a^b g = \int_a^b (h + ik) = \int_a^b h + i\int_a^b k. \qquad (6)$$

For example, the real and imaginary parts of the function

$$g : \theta \longmapsto \exp(i\theta) \qquad (\theta \in [a, b])$$

i.e.

$$g : \theta \longmapsto \cos\theta + i\sin\theta$$

are respectively

$$\theta \longmapsto \cos\theta \text{ and } \theta \longmapsto \sin\theta \qquad (\theta \in [a, b]).$$

Thus

$$\int_a^b \exp(i\theta)\, d\theta$$

means

$$\int_a^b \cos\theta\, d\theta + i\int_a^b \sin\theta\, d\theta.$$

$$= (\sin b - \sin a) + i(-\cos b + \cos a)$$

$$= \frac{\exp(ib) - \exp(ia)}{i}.$$

(iii) We originally defined $\mathfrak{L}[f]$ to have as domain some subset of R (sub-section 29.1.3 of *Unit 29*). Now we are extending the domain to include some complex numbers, such as iu with u real, as well.

From the point of view of this course, a major difference between Fourier and Laplace transforms is that to invert Fourier transforms we have a general method based on integrals such as

$$\frac{2}{\pi}\int_0^\infty F_c(y)\cos(xy)\, dy \qquad (7)$$

whereas to invert a Laplace transform we have had to use the special techniques and tables discussed in *Unit 29*. This difference is not as fundamental as it may look, because an inversion formula for the Laplace transform does exist; however, we shall not study this formula because it involves a type of integration which is outside the scope of this course.

Example

Consider the function (note the domain)

$$f : x \longmapsto e^{-ax} \qquad (x \in [0, \infty))$$

where a is some positive number. Note that f is not finitary, but even so the integrals defining F_c and F_s converge.

Working out the integrals that define F_c, F_s and $\mathfrak{L}[f]$, we obtain

$$F_c(y) = \int_0^\infty e^{-ax}\cos(xy)\, dx$$

$$= \frac{a}{a^2 + y^2} \qquad (y \in R),$$

$$F_s(y) = \int_0^\infty e^{-ax}\sin(xy)\, dx$$

$$= \frac{y}{a^2 + y^2} \qquad (y \in R),$$

and

$$\mathcal{L}[f](y) = \int_0^\infty e^{-ax}e^{-xy}\,dx$$

$$= \frac{1}{y+a} \qquad (y \in (-a, \infty))$$

Thus,

$$\mathcal{L}[f](-iy) = \frac{1}{-iy+a}$$

$$= \frac{iy+a}{(-iy+a)(iy+a)}$$

$$= \frac{a}{y^2+a^2} + i\frac{y}{y^2+a^2}$$

so that Formula (5),

$$\mathcal{L}[f](-iy) = F_c(y) + iF_s(y),$$

is verified.

Example

For the function

$$f : x \longmapsto e^{ax} \qquad (x \in [0, \infty))$$

with a again positive, we have

$$\mathcal{L}[f](y) = \frac{1}{y-a} \qquad (y \in (a, \infty))$$

but F_c and F_s are not defined, since the relevant integrals *diverge*. Indeed, the condition:

$$\int_0^\infty e^{ax}\,dx \text{ converges}$$

is not satisfied. We can still calculate the real and imaginary parts of

$$\mathcal{L}[f](-iy)$$

but we would *not* be justified in applying (5) and calling them $F_c(y)$ and $F_s(y)$ respectively.

Exercises

1. If

$$f : x \longmapsto 2ix - e^{3ix} \qquad (x \in R),$$

calculate

$$\int_0^{\pi/2} f(x)\,dx.$$

2. (i) Find the Laplace transform of the function

$$f : x \longmapsto \begin{Bmatrix} 1-x & \text{if} & 0 \leqslant x < 1 \\ 0 & \text{if} & 1 \leqslant x \end{Bmatrix} \qquad (x \in [0, \infty)).$$

(ii) Evaluate the real and imaginary parts of

$$\mathcal{L}[f](-iy).$$

Are we justified in calling these $F_c(y)$ and $F_s(y)$ respectively?

(iii) Check by comparing with the formula for the Fourier cosine transform of a function agreeing with f on $[0, \infty)$, which we calculated in sub-section 31.1.1.

Solutions

1.
$$\int_0^{\pi/2}(2ix - e^{3ix})\,dx$$
$$= \int_0^{\pi/2}[(-\cos 3x) + i(2x - \sin 3x)]\,dx$$
$$= \int_0^{\pi/2}[-\cos 3x]\,dx + i\int_0^{\pi/2}(2x - \sin 3x)\,dx$$
$$= \left[-\frac{\sin 3x}{3}\right]_0^{\pi/2} + \left[x^2 + \frac{\cos 3x}{3}\right]_0^{\pi/2}$$
$$= \frac{1}{3} + i\left(\frac{\pi^2}{4} - \frac{1}{3}\right)$$

2. (i) *Calculation of* $\mathfrak{L}[f](y)$

$$\int_0^\infty f(x)e^{-yx}\,dx = \int_0^1 (1-x)e^{-yx}\,dx$$
$$= \left[\frac{e^{-yx}}{-y} - \left(\frac{xe^{-yx}}{-y} - \frac{e^{-yx}}{y^2}\right)\right]_0^1$$
$$= \frac{e^{-y}}{y^2} + \frac{1}{y} - \frac{1}{y^2}$$

(ii) *Calculation of the real and imaginary parts of* $\mathfrak{L}[f](-iy)$

$$\mathfrak{L}[f](-iy) = \frac{e^{iy}}{(-iy)^2} + \frac{1}{-iy} - \frac{1}{(-iy)^2}$$
$$= \frac{\cos y + i\sin y}{-y^2} + \frac{i}{y} + \frac{1}{y^2}$$
$$= \frac{1-\cos y}{y^2} + i\left(\frac{1}{y} - \frac{\sin y}{y^2}\right)$$

Application of the formula $\mathfrak{L}[f] = F_c + iF_s$

We have

$$F_c(y) = \int_0^\infty f(x)\cos(xy)\,dx$$
$$= \int_0^1 (1-x)\cos(xy)\,dx$$

which converges since the end-points are both finite. The integral for $F_s(y)$ also converges for the same reason and so we may conclude that

$$F_c(y) = \frac{1-\cos y}{y^2}$$
$$F_s(y) = \frac{1}{y} - \frac{\sin y}{y^2}$$

(iii) *Comparison with direct calculation of* F_c

In sub-section 31.1.1 we calculated the Fourier transform of the function

$$x \longmapsto \begin{Bmatrix} 1-|x| & \text{if} & |x| < 1 \\ 0 & \text{if} & |x| \geqslant 1 \end{Bmatrix} \quad (x \in R)$$

Although this is not the same function as our f, its images agree with $f(x)$ for $x \geqslant 0$ and hence the transform $F_c(y)$ is the same for both functions. We obtained there, by direct integration,

$$F_c(y) = \frac{1-\cos y}{y^2}$$

which agrees with the result obtained above by the Laplace transform.

31.1.3 Complex Fourier Transforms

After our digression in the preceding sub-section about functions with domain $[0, \infty)$ we return to our main preoccupation, functions with domain R. We wish to remove the restriction made in sub-section 31.1.1 that the function f must be either even or odd, while retaining the useful symmetry in the formulas relating a function and its transform. More precisely, we have *different* definitions of the Fourier transform for functions with domain $[0, \infty)$ and R; they are related by defining, for f with domain $[0, \infty)$ an even and an odd extension (see pages K350–1); for example, the even extension ϕ is defined by

$$\phi: R \longmapsto R$$
$$\phi(x) = f(x) \quad \text{if} \quad x \geqslant 0$$
$$\phi(x) = \phi(-x) \quad \text{if} \quad x < 0.$$

Suppose, therefore, that f is a function that satisfies (i), (ii) and (iv), but is not necessarily odd or even. Since we already know how to deal with even and odd functions, we start by expressing f with domain R as the sum of an even and an odd function, using the method given in *Unit 22*, (Section 9-3 of K).

$$f = f_E + f_O$$

where

$$f_E(x) = \tfrac{1}{2}[f(x) + f(-x)]$$
$$f_O(x) = \tfrac{1}{2}[f(x) - f(-x)].$$

We can then calculate two transforms: the cosine transform of $f_E(x)$, and the sine transform of $f_O(x)$.

$$F_c(y) = \int_0^\infty f_E(x) \cos(xy)\, dx$$
$$F_s(y) = \int_0^\infty f_O(x) \sin(xy)\, dx$$

It would be possible to regard this pair of functions (F_c, F_s) as the "transform" of f, and we could use the inversion formulas from sub-section 31.1.1 to recover f_E, f_O, and hence f when this pair of functions was known; but this procedure is cumbersome and lacks the pleasing symmetry of the formulas we gave in sub-section 31.1.1 for the cases where f is even or odd. What we would like is a useful way of combining F_c and F_s into a single function in such a way that this new function is related in a symmetrical way to f.

A clue to the way to do it is provided by the preceding sub-section, where we saw that it is profitable to consider the combination $F_c + iF_s$. Let us try the same thing here. The combination we are interested in is

$$\begin{aligned}
&F_c(y) + iF_s(y)\\
&= \int_0^\infty f_E(x)\cos(xy)\, dx + i\int_0^\infty f_O(x)\sin(xy)\, dx\\
&= \int_0^\infty \left\{\frac{f(x)+f(-x)}{2}\cos(xy) + i\,\frac{f(x)-f(-x)}{2}\sin(xy)\right\} dx\\
&= \int_0^\infty \{\tfrac{1}{2}f(x)[\cos(xy) + i\sin(xy)]\\
&\qquad\qquad + \tfrac{1}{2}f(-x)[\cos(xy) - i\sin(xy)]\}dx\\
&= \int_0^\infty \{\tfrac{1}{2}f(x)e^{ixy} + \tfrac{1}{2}f(-x)e^{-ixy}\}\, dx \qquad \text{(by Euler's formula)}\\
&= \int_0^\infty (\text{even part of } [f(x)e^{ixy}])\, dx\\
&= \frac{1}{2}\int_{-\infty}^\infty f(x)e^{ixy}\, dx \qquad \text{(Equation (9-5), page K335).}
\end{aligned}$$

Once again we have an integral transform, differing somewhat from the others we have considered in having range of integration $(-\infty, \infty)$ in place of $[0, \infty)$ and in being complex for real y. We call

$$\frac{1}{2}\int_{-\infty}^{\infty} f(x)e^{ixy}\,dx$$

the *Fourier transform* of f.

We can find its inversion formula from those for the sine and cosine transforms as follows

$$\begin{aligned}
f(x) &= f_E(x) + f_O(x) \\
&= \frac{2}{\pi}\int_0^{\infty} F_c(y)\cos(xy)\,dy + \frac{2}{\pi}\int_0^{\infty} F_s(y)\sin(xy)\,dy \\
&= \frac{2}{\pi}\int_0^{\infty}\left(F_c(y)\frac{e^{ixy}+e^{-ixy}}{2} + F_s(y)\frac{e^{ixy}-e^{-ixy}}{2i}\right)dy
\end{aligned}$$

(by Euler's formula)

$$= \frac{2}{\pi}\int_0^{\infty} \tfrac{1}{2}e^{ixy}[F_c(y) - iF_s(y)] + \tfrac{1}{2}e^{-ixy}[F_c(y) + iF_s(y)]\,dy.$$

If we write $F(y) = F_c(y) + iF_s(y)$, this becomes

$$\begin{aligned}
f(x) &= \frac{2}{\pi}\int_0^{\infty}(\tfrac{1}{2}e^{ixy}F(-y) + \tfrac{1}{2}e^{-ixy}F(y))\,dy \\
&= \frac{1}{\pi}\int_0^{\infty} e^{-ixy}F(y)\,dy + \frac{1}{\pi}\int_{-\infty}^{0} e^{-ixy}F(y)\,dy \\
&= \frac{1}{\pi}\int_{-\infty}^{\infty} e^{-ixy}F(y)\,dy.
\end{aligned}$$

Thus for Fourier transforms, as for sine and cosine transforms, the inversion formula is very similar to the formula defining the transform itself.

It is usual to make the formulas even more symmetrical, by re-scaling the definition of the Fourier transform as

$$\tilde{f}(y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{ixy}\,dx$$

where

$$\tilde{f} = \sqrt{\frac{2}{\pi}}\,F.$$

The inversion formula is then

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \tilde{f}(y)e^{-ixy}\,dy.$$

These formulas are the central ones for us. The re-scaling has a role other than symmetry. If f belongs to a certain Euclidean space, the Fourier transform is *length preserving* with this scale. We discuss this further in Section 31.3.

As a further extension of these ideas, we would like to do away with the condition that f be finitary. It turns out that this can be done. A sufficient condition for the validity of the above inversion formula is that f be continuous and piecewise smooth and that f be *absolutely integrable*, by which we mean that the integral

$$\int_{-\infty}^{\infty} |f(x)|\,dx$$

is convergent. If f has a jump discontinuity at $x = a$, say, it can be shown that the inversion formula converges to

$$\frac{f(a^+) + f(a^-)}{2}.$$

provided f satisfies certain conditions. (Compare this with Theorem 9-1, page K340.)

Example

Find the Fourier transform $\tilde{f}$ of f, where

$$f(x) = e^{-a|x|} \qquad (x \in R)$$

and a is some positive number.

The Fourier transform is $\tilde{f}$, where

$$\begin{aligned}\tilde{f}(y) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a|x|} e^{ixy}\, dx \\ &= \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{\infty} e^{-a|x|} \cos(xy)\, dx + \frac{i}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a|x|} \sin(xy)\, dx \right] \\ &= \frac{2}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-a|x|} \cos(xy)\, dx,\end{aligned}$$

since

$$x \longmapsto e^{-a|x|} \cos(xy)$$

is even and

$$x \longmapsto e^{-a|x|} \sin(xy)$$

is odd. Using the result of the example in sub-section 31.1.2, we can evaluate the integral, obtaining

$$\tilde{f}(y) = \frac{2a}{\sqrt{2\pi}(a^2 + y^2)}.$$

Applying the inversion formula we obtain the integral

$$\frac{\pi}{a} e^{-a|x|} = \int_{-\infty}^{+\infty} \frac{1}{a^2 + y^2} e^{-ixy}\, dy$$

Exercises

1. Calculate the Fourier transform $\tilde{f}$ of f, where

   $$f(x) = e^{-a|x|} \sin bx \qquad (x \in R).$$

2. Show that if:

   $$f: R \longrightarrow R$$

   is even then the image set of $\tilde{f}$ is a subset of R.

3. If

   $$f: R \longrightarrow R$$

   is odd, describe the image set of $\tilde{f}$.

Solutions

1. $$f(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a|x|} \sin bx\, e^{ixy}\, dx$$
$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a|x|} \sin bx\, (\cos(xy) + i \sin(xy))\, dx$$
$$= \frac{2}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-a|x|} i \sin(bx) \sin(xy)\, dx,$$

(since the real part of the integrand is odd and the imaginary part is even)

$$= \frac{2i}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-a|x|} [\tfrac{1}{2}\cos(b-y)x - \tfrac{1}{2}\cos(b+y)x]\, dx$$
$$= \frac{i}{\sqrt{2\pi}} \left(\frac{a}{a^2 + (b-y)^2} - \frac{a}{a^2 + (b+y)^2} \right),$$

by the formula in the example of this sub-section.

2. $$\tilde{f}(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) (\cos(xy) + i \sin(xy))\, dx$$
$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \cos(xy)\, dx$$

(by Equation 9-6, page K335, since $f(x) \sin(xy)$ is odd). Since $\tilde{f}(y)$ is real, the image set of $\tilde{f}$ is a subset of R.

3. If f is odd, $\tilde{f}(y)$ is imaginary (i.e. its real part is the zero function for all $y \in R$). This is because

$$x \longmapsto f(x) \cos(xy)$$

is odd and hence

$$\int_{-\infty}^{\infty} f(x) \cos(xy)\, dx = 0.$$

The image set of $\tilde{f}$ is therefore a subset of the imaginary numbers $\{it : t \in R\}$.

31.1.4 Properties of Fourier Transforms

In formulating the definition and inversion formula for Fourier transforms, we have been led to define a new type of integral, in which the integrand is a complex-valued function (i.e. one whose codomain is C, the set of all complex numbers). For such functions we can integrate the real and imaginary parts separately, but the manipulation of such integrals is much simplified if we do not timidly convert them at once into integrals of real functions, but instead use complex numbers throughout. For example, the result we obtained in sub-section 31.1.2

$$\int_a^b e^{i\theta}\, d\theta = \frac{e^{ib} - e^{ia}}{i} \tag{1}$$

could have been obtained by assuming that the formula

$$\int_a^b e^{zx}\, dx = \left(\frac{1}{z}\right) [e^{zb} - e^{za}]$$

applies even when z is complex. For example, if F is defined by

$$F(\theta) = e^{i\theta} = \cos\theta + i \sin\theta$$

we have

$$F'(\theta) = -\sin\theta + i\cos\theta$$
$$= ie^{i\theta}$$

as we might expect from the formula

$$\frac{d}{d\theta}(e^{a\theta}) = ae^{a\theta}$$

for real a. We also have

$$\int_a^b ie^{i\theta}\,d\theta = e^{ib} - e^{ia}$$

and the result (1) follows on dividing both sides by i.

Since the fundamental theorem applies to such complex integrals, the rules of integration derived from it apply too. In particular, we may use integration by substitution and integration by parts for complex integrals just as for real ones. There is one caution, however: substitutions must always be done with real-valued functions (see Example 1 below). This will ensure that the new integrands have domain R

Example 1 (integration by substitution)
Given that the Fourier transform of f is $\tilde{f}$, what is the Fourier transform of the function

$$x \longmapsto f(x-a) \qquad (x \in R)$$

where a is a real constant?

We are given

$$\tilde{f}(y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\,e^{ixy}\,dx$$

and we want

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x-a)\,e^{ixy}\,dx.$$

We make the substitution $x = z + a$ obtaining, since $\dfrac{dx}{dz} = 1$,

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(z)\,e^{i(z+a)y}\,dz \tag{2}$$

The integral (2) can be written

$$= \frac{e^{iay}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(z)e^{izy}\,dz$$
$$= e^{iay}\tilde{f}(y).$$

The Fourier transform of

$$x \longmapsto f(x-a)$$

is therefore

$$y \longmapsto e^{iay}\tilde{f}(y).$$

This is the analogue for Fourier transforms of the second shifting theorem for Laplace transforms (page K195), which states that if a is positive and if f is defined by

$$f(t) = \begin{cases} g(t-a) & \text{if } t > a \\ 0 & \text{if } t < a \end{cases}$$

then

$$\mathfrak{L}[f] - e^{-as}\mathfrak{L}[g].$$

Notice, however, that if a had not been real the method would have broken down; in that case $z = x - a$ would have been complex, so that the transformed integrand in (2) would have had a complex domain, and we have not discussed the integration of functions with complex domain.

Example 2 (integration by parts)

Given that the Fourier transform of f is $\tilde{f}$, find that of its derived function f'. (Assume that the derived function f' has a Fourier transform and that $f(x)$ approaches 0 for large positive and large negative x.)
The Fourier transform of f' is $\tilde{f'}$ where

$$\tilde{f'}(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f'(x)\, e^{ixy}\, dx.$$

Integration by parts gives

$$\tilde{f'}(y) = \left[\frac{1}{\sqrt{2\pi}} f(x) e^{ixy}\right]_{-\infty}^{\infty} - \frac{i}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) y e^{ixy}\, dx \qquad (3)$$

where $[\phi(x)]_{-\infty}^{\infty}$ means, for any function ϕ with domain R and codomain C,

$$\lim_{x \to \infty} \phi(x) - \lim_{x \to \infty} \phi(-x),$$

provided both limits exist. Since we are assuming that $f(x)$ approaches 0 for large positive and large negative x, (3) reduces to

$$\tilde{f'}(y) = -\frac{i}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) y e^{ixy}\, dx$$

and so the Fourier transform of f' is

$$\tilde{f'} : y \longmapsto -iy\tilde{f}(y)$$

(What commutative diagram may be drawn here?)

There is another technique of integration which is useful for Fourier transforms, called *differentiation under the integral sign*. We introduce it now merely because we have not had occasion to use it before, and not because of any logical connection with Fourier integrals or complex-valued functions. The rule states that, under certain conditions, the differentiation linear transformation with respect to one variable commutes with the integration linear transformation with respect to another:

$$\frac{d}{dy} \int_a^b \phi(x, y)\, dx = \int_a^b \frac{\partial \phi}{\partial y}(x, y)\, dx \qquad (4)$$

where ϕ is a suitable function from $R \times R$ to R.

Example 3

If $\phi(x, y) = e^{xy}$, then the left-hand side of Equation (4) is

$$\frac{d}{dy} \int_a^b e^{xy}\, dx = \frac{d}{dy}\left(\frac{e^{by} - e^{ay}}{y}\right)$$
$$= \frac{be^{by} - ae^{ay}}{y} - \frac{e^{by} - e^{ay}}{y^2}.$$

The right-hand side of (4) is

$$\int_a^b \frac{\partial}{\partial y}(e^{xy})\, dx = \int_a^b xe^{xy}\, dx$$
$$= \left[\frac{xe^{xy}}{y} - \frac{e^{xy}}{y^2}\right]_a^b \text{ (by Section III.5.2 of TI)}$$
$$= \frac{be^{by} - ae^{ay}}{y} - \frac{e^{by} - e^{ay}}{y^2}$$

as before.

The formula (4) for differentiation under the integral sign may be applied whenever the functions ϕ and $\frac{\partial \phi}{\partial y}$ are continuous. (See Theorem I-35 on page K670.) If the interval of integration is infinite, then the formula is the same:

$$\frac{d}{dy}\int_{-\infty}^{\infty} \phi(x, y)\, dx = \int_{-\infty}^{\infty} \frac{\partial \phi}{\partial y}(x, y)\, dx,$$

but the conditions for it to be valid are more complicated: it is sufficient that ϕ and $\frac{\partial \phi}{\partial y}$ be continuous; that there exist functions ϕ_1 and ϕ_2 such that

$$|\phi(x, y)| < \phi_1(x) \text{ and } \left|\frac{\partial \phi}{\partial y}(x, y)\right| < \phi_2(x)$$

for all y; and that

$$\int_{-\infty}^{\infty} \phi_1(x)\, dx \text{ and } \int_{-\infty}^{\infty} \phi_2(x)\, dx$$

both converge. A proof is given in Theorem I-41 on page K676, but you are not expected to study it.

Example 4

Given that the Fourier transform of f is $\tilde{f}$, what is the Fourier transform of

$$g : x \longmapsto xf(x) \qquad (x \in R)?$$

(It is assumed, of course, that both f and g have Fourier transforms.) The required Fourier transform is

$$\begin{aligned}
\tilde{g}(y) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} xf(x)e^{ixy}\, dx \\
&= -\frac{i}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\frac{\partial}{\partial y}(e^{ixy})\, dx \\
&= -\frac{i}{\sqrt{2\pi}}\frac{d}{dy}\int_{-\infty}^{\infty} f(x)e^{ixy}\, dx \\
&= -i\frac{d}{dy}\tilde{f}(y) \\
&= -i(\tilde{f})'(y).
\end{aligned}$$

These four rules: the fundamental theorem of calculus, integration by substitution, integration by parts, and differentiation under the integral sign, provide the basis for the manipulations of complex Fourier transforms that we shall be performing in the rest of this unit. We have stated conditions for their validity, but they are often difficult to verify in practice. We may note that there are other conditions, not equivalent to the ones we have given. The point is that the conditions are related to the *space* the function belongs to. What is sometimes confusing is that the same function may belong to *different* spaces! We examine Euclidean spaces in Section 31.3.

Example 5

The Fourier transforms of certain functions are particularly useful in applications. One that is especially so is the function

$$f : x \longmapsto \exp(-\tfrac{1}{2}x^2) \qquad (x \in R)$$

associated with Gauss. To find its Fourier transform we evaluate the integral

$$\tilde{f}(y)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\exp\left(-\tfrac{1}{2}x^2\right)e^{ixy}\,dx. \tag{5}$$

We can do this by means of the following device. We differentiate both sides of (5) with respect to y, using the rule of differentiation under the integral sign, which is valid in this case. This gives

$$\begin{aligned}(\tilde{f})'(y)&=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\exp\left(-\tfrac{1}{2}x^2\right)\frac{\partial}{\partial y}\left(e^{ixy}\right)dx\\&=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\exp\left(-\tfrac{1}{2}x^2\right)ixe^{ixy}\,dx\\&=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{d}{dx}\left(-i\exp\left(-\tfrac{1}{2}x^2\right)\right)e^{ixy}\,dx\end{aligned}$$

Integration by parts gives

$$\begin{aligned}(\tilde{f})'(y)&=\frac{1}{\sqrt{2\pi}}\left\{\left[-i\exp\left(-\tfrac{1}{2}x^2\right)e^{ixy}\right]_{-\infty}^{\infty}\right.\\&\qquad\left.+\,i\int_{-\infty}^{\infty}\exp\left(-\tfrac{1}{2}x^2\right)\frac{\partial}{\partial x}\left(e^{ixy}\right)dx\right\}\\&=0-\frac{y}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\exp\left(-\tfrac{1}{2}x^2\right)e^{ixy}\,dx\\&=-y\tilde{f}(y).\end{aligned}$$

Thus we have obtained a differential equation for $\tilde{f}$. Solving it by the method described in *Unit 4, Differential Equations I*, (page K96) we obtain

$$\tilde{f}(y)=A\exp\left(-\tfrac{1}{2}y^2\right) \tag{6}$$

where A is some real number.

We can evaluate A by using the Fourier inversion formula, which gives

$$\begin{aligned}f(x)&=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\tilde{f}(y)e^{-ixy}\,dy\\&=A\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\exp\left(-\tfrac{1}{2}y^2\right)e^{-iyx}\,dy\\&=A\tilde{f}(-x)\qquad\text{by (5)}\\&=A^2\exp\left(-\tfrac{1}{2}x^2\right)\qquad\text{by (6)}\end{aligned}$$

Thus we can conclude that $A=\pm1$. Trying the special value $y=0$ in (5) gives

$$\tilde{f}(0)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\exp\left(-\tfrac{1}{2}x^2\right)dx>0$$

and so, by (6), A must be positive. It is therefore $+1$, and we conclude that the function

$$x\longmapsto\exp\left(-\tfrac{1}{2}x^2\right)$$

is its own Fourier transform!

Exercises

1. Use the result

$$\tilde{f}(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-\tfrac{1}{2}x^2\right) e^{ixy}\, dx = \exp\left(-\tfrac{1}{2}y^2\right)$$

in Example 5 to evaluate

$$\int_{-\infty}^{\infty} \exp\left(-\tfrac{1}{2}x^2\right) dx.$$

2. If g and f are related by

$$g(x) = f(ax) \qquad (x \in R)$$

with a some given positive number, how are $\tilde{g}$ and $\tilde{f}$ related? (*Hint:* look at Example 1 again.)

3. Use the results of Examples 1 and 5, and Exercise 2, to find the Fourier transform of

$$f : x \longmapsto \exp\left(-\tfrac{1}{2}c(x-b)^2\right) \qquad (x \in R)$$

where c and b are given real numbers, with $c > 0$.

4. Use differentiation under the integral sign to derive from

$$\int_0^x e^{-st}\, dt = \frac{1 - e^{-sx}}{s},$$

formulas for

$$\text{(i)} \int_0^x te^{-st}\, dt \quad \text{and} \quad \text{(ii)} \int_0^x t^2 e^{-st}\, dt$$

(*Hint:* what is $\dfrac{\partial}{\partial s}(e^{-st})$?)

Solutions

1. If $f(x) = \exp\left(-\tfrac{1}{2}x^2\right)$, we have

$$\int_{-\infty}^{\infty} \exp\left(-\tfrac{1}{2}x^2\right) dx = \sqrt{2\pi}\, \tilde{f}(0) = \sqrt{2\pi}.$$

2. By definition

$$\tilde{g}(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(ax) e^{iyx}\, dx.$$

Using integration by substitution with $x = \dfrac{z}{a}$,

$$\tilde{g}(y) = \frac{1}{a} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(z) e^{(iyz)/a}\, dz$$

$$= \frac{1}{a} \tilde{f}\left(\frac{y}{a}\right) \text{ by definition.}$$

3. We have

$$f(x) = g(x - b)$$

where

$$g : u \longmapsto \exp\left(-\tfrac{1}{2}cu^2\right)$$

and

$$g(u) = h(\sqrt{c}u)$$

where

$$h : v \longmapsto \exp\left(-\tfrac{1}{2}v^2\right)$$

We must have

$$\tilde{f}(y) = e^{iyb}\tilde{g}(y) \qquad \text{(Example 1)}$$

and

$$\tilde{g}(y) = \frac{1}{\sqrt{c}}\tilde{h}\left(\frac{y}{\sqrt{c}}\right) \qquad \text{(Example 5 and Exercise 2)}$$

and so

$$\tilde{f}(y) = e^{iyb}\frac{1}{\sqrt{c}}\exp\left[-\tfrac{1}{2}\left(\frac{y}{\sqrt{c}}\right)^2\right]$$

$$= \frac{1}{\sqrt{c}}\exp\left(-\frac{y^2}{2c} + iyb\right).$$

4. (i)

$$\int_0^x te^{-st}\,dt = -\int_0^x \left(\frac{\partial}{\partial s}e^{-st}\right)dt$$

$$= -\frac{d}{ds}\int_0^x e^{-st}\,dt$$

$$= -\frac{d}{ds}\left(\frac{1-e^{-sx}}{s}\right)$$

$$= \frac{1-e^{-sx}}{s^2} - \frac{xe^{-sx}}{s}.$$

(ii)

$$\int_0^x t^2e^{-st}\,dt = \int_0^x \frac{\partial^2}{\partial s^2}e^{-st}\,dt$$

$$= \frac{d}{ds}\int_0^x \frac{\partial}{\partial s}e^{-st}\,dt$$

$$= -\frac{d}{ds}\left(\frac{1-e^{-sx}}{s^2} - \frac{xe^{-sx}}{s}\right)$$

$$= \frac{2-2e^{-sx}}{s^3} - \frac{2xe^{-sx}}{s^2} - \frac{x^2e^{-sx}}{s}.$$

31.1.5 Summary of Section 31.1

In this section we defined the terms

finitary function	(page C6)	* *
Fourier cosine transform	(page C7)	* *
inversion formula	(page C8)	* *
Fourier sine transform	(page C8)	* *
integral transform	(page C10)	*
Fourier transform	(page C16)	* *
absolutely integrable function	(page C16)	* *

Notation

F_c (page C7)
F_s (page C8)
$\tilde{f}$ (page C16)

Techniques

1. Integrate functions from R to C by applying the rule * *

$$\int_a^b g = \int_a^b h + ik = \int_a^b h + i\int_a^b k$$

where $g = h + ik$.

2. Apply the following techniques of integration to functions from R to C: * *
 (i) integration by substitution,
 (ii) integration by parts,
 (iii) differentiation under the integral sign.

31.2 APPLICATION TO DIFFERENTIAL EQUATIONS

31.2.1 The Method

The application we shall make of Fourier transforms in this course is to the solution of partial differential equations such as the one-dimensional wave equation (*Unit 23*)

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{a^2}\frac{\partial^2 u}{\partial t^2} \tag{1}$$

and the one-dimensional heat conduction equation (*Unit 32*)

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{a^2}\frac{\partial u}{\partial t}$$

when the space variable x is not restricted to a finite interval. In this unit we shall apply the method to the wave equation, obtaining D'Alembert's solution which was given in sub-section 23.4.1 of *Unit 23*.

We shall also show how the method can yield new results, by applying it to the heat conduction equation.

To solve the wave equation by Fourier transforms we start by taking Fourier transforms of both sides of Equation (1) with respect to x; that is, we write

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{\partial^2 u}{\partial x^2}(x, t)\, e^{ixy}\, dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{1}{a^2}\frac{\partial^2 u}{\partial t^2}(x, t) e^{ixy}\, dx \tag{2}$$

To clarify the manipulations that follow we introduce some auxiliary notation. For each real number t we define the function of one real variable

$$\phi_t : x \longmapsto u(x, t) \qquad (x \in R). \tag{3}$$

For example, if $t = 2$ we have the function

$$\phi_2 : x \longmapsto u(x, 2) \qquad (x \in R).$$

The reason for this definition is that we are applying the Fourier transform to x only: x and t are treated differently. Using this notation the left-hand side of (2) becomes

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \phi_t''(x) e^{ixy}\, dx$$

the Fourier transform of ϕ_t''. Now we saw in Example 2 of sub-section 31.1.4 how to show, using integration by parts, that the Fourier transform of ϕ_t' is

$$y \longmapsto -iy\tilde{\phi}_t(y)$$

where

$$\tilde{\phi}_t : y \longmapsto \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \phi_t(x) e^{ixy}\, dx$$

is the Fourier transform of ϕ_t. By applying the same method a second time, we see that the transform of ϕ_t'' is

$$y \longmapsto (-iy)^2\tilde{\phi}_t(y) = -y^2\tilde{\phi}_t(y)$$

and so the left-hand side of (2) is $-y^2\tilde{\phi}_t(y)$.

To simplify the right-hand side of (2) we use a similar idea. We define functions, one for each real number y, by

$$\psi_y : t \longmapsto \tilde{\phi}_t(y) \qquad (t \in R) \tag{4}$$

So that

$$\psi_y(t) = \tilde{\phi}_t(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x, t)e^{ixy}\, dx.$$

The derivative of this function is $\psi_y'(t)$, where

$$\psi_y'(t) = \frac{d}{dt}\left\{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x, t)e^{ixy}\, dx\right\}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\partial u}{\partial t}(x, t)e^{ixy}\, dx$$

assuming that the conditions for differentiating under the integral sign are satisfied. In a similar way we obtain

$$\psi_y''(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\partial^2 u}{\partial t^2}(x, t)e^{ixy}\, dx.$$

Therefore

$$\frac{1}{a^2}\psi_y''(t) = \text{right-hand side of (2).}$$

Using our new forms for both the left- and the right-hand side, Equation (2) now becomes

$$-a^2y^2\tilde{\phi}_t(y) = \psi_y''(t).$$

Using (4) we can write this as

$$-a^2y^2\psi_y(t) = \psi_y''(t) \qquad (5)$$

Thus we have reduced the partial differential equation in u, Equation (1), to a whole set of ordinary differential equations in the functions ψ_y, one for each real number y. These functions are related to u by

$$\psi_y(t) = \tilde{\phi}_t(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x, t)e^{ixy}\, dx.$$

Taking the transform has replaced the double differentiation with respect to x by a multiplication by $-y^2$.
The ordinary differential equations (5) are easily solved. We know from *Unit 9, Differential Equations II*, that the functions

$$t \longmapsto \sin ayt$$

and

$$t \longmapsto \cos ayt$$

form a basis for the solution space (this is true even though the solution space is now a *complex* vector space).
Since we are interested in complex-valued functions, it is more convenient to use the alternative basis

$$\{t \longmapsto e^{iayt},\ t \longmapsto e^{-iayt}\}$$

and so the general solution of (5) may be written

$$\psi_y(t) = G_y e^{iayt} + F_y e^{-iayt} \qquad (t \in R)$$

where G_y and F_y are arbitrary constants. For each real number y we may choose G_y and F_y arbitrarily and still obtain a solution of (5). To describe the way G_y and F_y depend on y, let us introduce two new functions

$$\tilde{f}: y \longmapsto F_y \qquad (y \in R)$$

$$\tilde{g}: y \longmapsto G_y \qquad (y \in R)$$

The notation prejudges the issue: we shall be interested in the inverse Fourier transforms of the two complex-valued functions $\tilde{f}$ and $\tilde{g}$ which are at present more or less arbitrary.

We now want to use our knowledge of ψ_y to find u. Equation (4) tells us that

$$\psi_y(t) = \tilde{\phi}_t(y),$$

so that

$$\tilde{\phi}_t(y) = \tilde{g}(y)e^{iayt} + \tilde{f}(y)e^{-iayt} \qquad (y \in R) \tag{6}$$

holds for every t.

The inverse Fourier transform of $\tilde{\phi}_t$ is $u(x, t)$, so that

$$\begin{aligned} u(x, t) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ixy}\tilde{\phi}_t(y)\, dy \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ixy}[e^{iayt}\tilde{g}(y) + e^{-iayt}\tilde{f}(y)]\, dy \qquad \text{by (6)} \end{aligned}$$

We can rewrite this as

$$\begin{aligned} u(x, t) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i(x-at)y}\, \tilde{g}(y)\, dy \\ &\quad + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i(x+at)y}\, \tilde{f}(y)\, dy \qquad (7) \\ &= g(x-at) + f(x+at) \qquad (8) \end{aligned}$$

where g and f are the inverse Fourier transforms of $\tilde{g}$ and $\tilde{f}$. Since $\tilde{g}$ and $\tilde{f}$ are more or less arbitrary, so are g and f—except, of course, that we would like them both to be real so as to make $u(x, t)$ real. Equation (8) is just D'Alembert's solution of the wave equation, which we studied in sub-section 23.4.1 of *Unit 23*. But as a concomitant of all our work, we also have Equation (7), which is often useful.

Exercises

1. Describe briefly the *main* steps used in the Fourier transform method of solution of a partial differential equation.
2. Using the Fourier transform method obtain a solution for the heat conduction equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{a^2}\frac{\partial u}{\partial t}$$

where

$$u : R \times [0, \infty) \longrightarrow R$$

and $a > 0$. (Assume that u and $\dfrac{\partial u}{\partial x}$ approach zero for large positive and negative x, and any other existence condition you need.) Give your answer in the form of an integral involving an unspecified function $\tilde{g}$, analogous to Equation (7) above. Check your final answer by substituting in the above differential equation.

3. Show how the unspecified function $\tilde{g}$ in the previous solution can be determined from a knowledge of the function

$$x \longmapsto u(x, 0).$$

Solutions

1. The main steps are:
   (i) take Fourier transform of the partial differential equation;
   (ii) by defining appropriate functions of one variable, write down an equivalent ordinary differential equation;
   (iii) solve the ordinary differential equation;

(iv) generalize to functions the arbitrary constant(s) in this solution and use it to obtain the solution of the partial differential equation by applying the inverse Fourier transform.

$$\begin{tikzcd}
\text{P.D.E.} \arrow[r] \arrow[d] & \text{O.D.E.} \arrow[d] \\
\text{Solution} & \text{Solution} \arrow[l]
\end{tikzcd}$$

2. The transformed equation is

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ixy}\frac{\partial^2 u}{\partial x^2}(x, t)\,dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ixy}\frac{1}{a^2}\frac{\partial u}{\partial t}(x, t)\,dx.$$

The equation analogous to (5) is (using the same notation)

$$-y^2\psi_y = \frac{1}{a^2}\psi_y'$$

which holds for every real number y. Solving this as an ordinary differential equation gives

$$\psi_y(t) = \tilde{\phi}_t(y) = c_y \exp(-a^2y^2t)$$

and since the arbitrary constant c_y may be different for different y the general solution of the differential equation for $\tilde{\phi}_t$ is

$$\tilde{\phi}_t(y) = \tilde{g}(y)\exp(-a^2y^2t)$$

where $\tilde{g}$ is an arbitrary function from R to C. Taking the inverse Fourier transform, we obtain the general solution

$$u(x, t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\tilde{g}(y)\exp(-a^2y^2t - ixy)\,dy.$$

For the time being, this cannot be simplified further.
Check: differentiation under the integral sign gives

$$\frac{\partial^2 u}{\partial x^2}(x, t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\tilde{g}(y)\exp(-a^2y^2t - ixy)(-y^2)\,dy$$

$$\frac{\partial u}{\partial t}(x, t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\tilde{g}(y)\exp(-a^2y^2t - ixy)(-a^2y^2)\,dy$$

so that

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{a^2}\frac{\partial u}{\partial t},$$

as required.

3. The previous result, with $t = 0$, gives

$$u(x, 0) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\tilde{g}(y)e^{-ixy}\,dy$$

Thus $u(x, 0)$ is the inverse Fourier transform of $\tilde{g}$ and hence $\tilde{g}$ is the Fourier transform of $x \longmapsto u(x, 0)$. The required formula is therefore

$$\tilde{g}(y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} u(x, 0)e^{ixy}\,dx.$$

31.2.2 Repeated Integrals

Both the Fourier transformation and its inverse mapping are linear transformations (see the exercise on page C11): the two diagrams below are commutative.

$$\begin{tikzcd}
(f, g) \arrow[r, "+"] \arrow[d] & f+g \arrow[d] \\
(\tilde{f}, \tilde{g}) \arrow[r, "+"'] & \tilde{f}+\tilde{g}
\end{tikzcd}$$

$$\begin{tikzcd}
f \arrow[r, "\text{scalar multiplication}"] \arrow[d] & \lambda f \arrow[d] \\
\tilde{f} \arrow[r, "\text{scalar multiplication}"'] & \lambda \tilde{f}
\end{tikzcd}$$

On the other hand, taking Fourier transforms is *not* compatible with multiplication of functions.

But is there a binary operation $\square$ on the space of functions so that

$$\begin{tikzcd}
(f, g) \arrow[r, "\times"] \arrow[d] & f \times g \arrow[d] \\
(\tilde{f}, \tilde{g}) \arrow[r, "\square"'] & \tilde{f} \square \tilde{g} = (f \times g)^{\sim}
\end{tikzcd}$$

does commute? If so, finding $\square$ would enable us to express $(f \times g)^{\sim}$ in terms of $\tilde{f}$ and $\tilde{g}$ above.

Useful as this would be, finding a binary operation $*$ such that

$$\begin{tikzcd}
(f, g) \arrow[r, "*"] \arrow[d] & f * g \arrow[d] \\
(\tilde{f}, \tilde{g}) \arrow[r, "\times"'] & \tilde{f} \times \tilde{g} = (f * g)^{\sim}
\end{tikzcd}$$

commutes would be even more useful. For we could then exploit our knowledge of f and g. For example, from Exercise 2 of sub-section 31.2.1 we have the solution of the heat conduction equation in the form

$$u(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{g}(y) \exp(-a^2 y^2 t - ixy)\, dy.$$

If

$$\tilde{f}_t : y \longmapsto \exp(-a^2 y^2 t) \qquad (y \in R),$$

then

$$u(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{g}(y) \times \tilde{f}_t(y) e^{-ixy}\, dy;$$

and if the inverse transforms of $\tilde{g}$ and $\tilde{f}_t$, g and f_t, are known, and if $*$ exists, then

$$u(x, t) = g(x) * f_t(x).$$

In this sub-section we shall find the answer to this problem, namely to find h in terms of f and g if we know that $\tilde{h} = \tilde{f} \times \tilde{g}$. That is, we have

$$h(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{f}(y)\tilde{g}(y) e^{-ixy}\, dy \tag{1}$$

as the input datum. But we may replace $\tilde{f}$ in this:

$$h(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left\{ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(z) e^{izy}\, dz \right\} \tilde{g}(y) e^{-ixy}\, dy \tag{2}$$

The right-hand side is the integral of a function which is itself defined, as an integral. Such an expression is usually called a *repeated integral.*

An important property of repeated integrals is that we can usually reverse the order in which the integrations are done. (This reversal of order has analogues in situations we have seen already. For example for an $n \times m$ matrix

$$\begin{bmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nm} \end{bmatrix}$$

we can interchange the order of summations over rows and over columns·

$$\sum_{i=1}^{n} \sum_{j=1}^{m} a_{ij} = \sum_{j=1}^{m} \sum_{i=1}^{n} a_{ij}$$

since we can calculate the sum of all the elements in the matrix either by calculating the n row sums and then adding them, or by calculating the m column sums and then adding them.) For finite ranges of integration such interchanges

$$\int_a^b \left\{ \int_c^d F(x, y)\, dy \right\} dx = \int_c^d \left\{ \int_a^b F(x, y)\, dx \right\} dy \tag{3}$$

are always valid provided only that the integrand is continuous. For infinite integrals

$$\int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} F(x, y)\, dy \right\} dx = \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} F(x, y)\, dx \right\} dy \tag{3'}$$

they are valid if F is continuous and if

$$\int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} |F(x, y)|\, dy \right\} dx$$

converges. (A related result is Theorem I-40 on page K675.)

Assuming that these conditions are satisfied by the integrand in the repeated integral (2), i.e. by

$$f(z)\tilde{g}(y)e^{-iy(x-z)}$$

for each $x \in R$, we may use (3)′ and obtain

$$h(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(z) \left\{ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{g}(y) e^{-iy(x-z)}\, dy \right\} dz$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(z) g(x-z)\, dz \tag{4}$$

The change of variable $u = x - z$ yields an alternative way of writing the same result

$$h(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-u) g(u)\, du \tag{5}$$

It is conventional to use the same variable of integration in (4) and (5). Hence (5) becomes

$$h(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-z) g(z)\, dz \tag{6}$$

The binary operation $*$ is defined by (4) or (6).

$$f * g(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(z) g(x-z)\, dz = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-z) g(z)\, dz.$$

This is an important result called the

Convolution Theorem

If

$$\tilde{h} = \tilde{f} \times \tilde{g}$$

then

$$h = f * g.$$

h is called the *convolution* of f and g.

Exercise

In Exercise 2 of sub-section 31.2.1 we arrived at a solution of the heat conduction equation:

$$u(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{g}(y) \exp(-a^2 y^2 t) e^{-ixy} \, dy.$$

From Exercise 3 of sub-section 31.1.4, the Fourier transform of

$$y \longmapsto \exp(-a^2 y^2 t)$$

is

$$\phi_t : x \longmapsto \frac{1}{a\sqrt{2t}} \exp\left(-\frac{x^2}{4a^2 t}\right)$$

for each $t \in R$.

Verify by differentiation that

$$u = g * \phi_t$$

$\left(\text{i.e. } g * \phi_t \text{ satisfies } \dfrac{\partial^2 u}{\partial x^2} = \dfrac{1}{a^2}\dfrac{\partial u}{\partial t}\right)$

Solution

Using differentiation under the integral sign we obtain

$$\begin{aligned}
\frac{\partial u}{\partial t}(x, t) &= \frac{\partial}{\partial t} \int_{-\infty}^{\infty} g(z) \frac{1}{a\sqrt{2t}} \exp\left(-\frac{(z-x)^2}{4a^2 t}\right) dz \\
&= \int_{-\infty}^{\infty} g(z) \frac{\partial}{\partial t}\left(\frac{1}{a\sqrt{2t}} \exp\left(-\frac{(z-x)^2}{4a^2 t}\right)\right) dz \\
&= \int_{-\infty}^{\infty} g(z) \left(\frac{-1}{2\sqrt{2}at^{3/2}} + \frac{1}{a\sqrt{2t}} \frac{(z-x)^2}{4a^2 t^2}\right) \\
&\qquad\qquad \times \exp\left(-\frac{(z-x)^2}{4a^2 t}\right) dz
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial^2 u}{dx^2}(x, t) &= \frac{\partial^2}{\partial x^2} \int_{-\infty}^{\infty} g(z) \frac{1}{a\sqrt{2t}} \exp\left(-\frac{(z-x)^2}{4a^2 t}\right) dz \\
&= \int_{-\infty}^{\infty} g(z) \frac{1}{a\sqrt{2t}} \left(\frac{\partial^2}{\partial x^2} \exp\left(-\frac{(z-x)^2}{4a^2 t}\right)\right) dz \\
&= \int_{-\infty}^{\infty} g(z) \frac{1}{a\sqrt{2t}} \left(\frac{\partial}{\partial x}\left(\frac{z-x}{2a^2 t}\right.\right. \\
&\qquad\qquad \left.\left.\times \exp\left(-\frac{(z-x)^2}{4a^2 t}\right)\right)\right) dz \\
&= \int_{-\infty}^{\infty} g(z) \frac{1}{a\sqrt{2t}} \left(-\frac{1}{2a^2 t} + \frac{(z-x)^2}{4a^4 t^2}\right) \\
&\qquad\qquad \times \exp\left(-\frac{(z-x)^2}{4a^2 t}\right) dz \\
&= \frac{1}{a^2} \frac{\partial u}{\partial t}(x, t) \text{ as required.}
\end{aligned}$$

31.2.3 Summary of Section 31.2

In this section we defined the terms

repeated integral	(page C30)	* *
convolution of two functions	(page C31)	*

Theorem

(Convolution Theorem, page C31)

If * * *

$$\hat{h} = \hat{f} \times \hat{g}$$

then

$$h = f * g$$

where

$$f * g(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(z)g(x-z)\,dz = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-z)g(z)\,dz$$

Techniques

1. Use the Fourier transform to solve partial differential equations in two variables, in particular the one-dimensional wave equation and heat conduction equations. * * *
2. Reverse repeated infinite integrals. * *

Notation

ϕ_t	(page C25)
ψ_y	(page C25)
$f * g$	(page C30)

31.3 FOURIER TRANSFORMS AND EUCLIDEAN SPACES

31.3.1 Fourier Transforms and Euclidean Spaces

To round off this discussion we shall show how Fourier transforms are related to the Euclidean space theory we were studying when we first came across the idea of a Fourier coefficient (*Unit 19, Least-squares Approximation*, pages K286–7). Since we have only considered real Euclidean spaces we shall confine the discussion mainly to the types of Fourier transforms which are guaranteed to be real, the sine and cosine transforms F_s and F_c defined in sub-section 31.1.1. To make the relation between a function and its transform symmetric we shall modify the factor $2/\pi$ which appeared in the inversion formula for our previous definition. That is, we re-define the cosine transform of f to be

$$\tilde{f}_c(y) = \sqrt{\frac{2}{\pi}}\, F_c(y) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos(xy)\, dx \qquad (y \in R)$$

so that the inversion formula is

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \tilde{f}_c(y) \cos(xy)\, dy \qquad (x \in [0, \infty))$$

A set of conditions under which this pair of formulas is valid is: (i) f has domain $[0, \infty)$ and codomain R; (ii) f is continuous and f' and f'' are piecewise continuous; (iii) f is absolutely integrable on $[0, \infty)$, i.e. $\int_0^\infty |f(x)|\, dx$ converges. The set of all functions with these properties forms a vector space which we shall call $V[0, \infty)$. We can attempt to convert this into a Euclidean space in the same way as we did for the other vector spaces of functions we have dealt with, by defining an inner product on it. The obvious definition to use is

$$\mathbf{f} \cdot \mathbf{g} = \int_0^\infty fg$$

with the associated Euclidean norm $\|\mathbf{f}\|$, where

$$\|\mathbf{f}\|^2 = \mathbf{f} \cdot \mathbf{f} = \int_0^\infty f^2.$$

Before proceeding, we must ask whether this definition really does give us a Euclidean space. First of all, does $\mathbf{f} \cdot \mathbf{g}$ exist for every f, g in $V[0, \infty)$? The answer is, it does not: there are functions in $V[0, \infty)$ for which $\int_0^\infty f^2$ does not exist. An example is given in Appendix 2. It follows that $V[0, \infty)$ with the above norm is not a Euclidean space. A similar situation arose in *Unit 20, Euclidean Spaces II*, where we wanted to define a Euclidean space structure on the vector space R^∞ consisting of all infinite sequences of real numbers. We can deal with the problem in the same way as we did there: we define our Euclidean space to be the subspace of $V[0, \infty)$ for which

$$\int_0^\infty f^2$$

does converge.

Let us call this space $E[0, \infty)$. It can be verified that the axioms for a vector space are satisfied, so that $E[0, \infty)$ is a vector subspace of $V[0, \infty)$, and that $\mathbf{f} \cdot \mathbf{g}$ exists for all f, g in $E[0, \infty)$. Thus $E[0, \infty)$ is a Euclidean space.

The question naturally arises now whether the Fourier cosine transform (i.e. the linear transformation of $E[0, \infty)$ to itself

$$f \longmapsto \tilde{f}_c)$$

is a Euclidean space morphism; that is, whether $\mathbf{f} \cdot \mathbf{g} = \tilde{\mathbf{f}}_c \cdot \tilde{\mathbf{g}}_c$ so that the diagram

$$\begin{tikzcd}
\mathbf{f}, \mathbf{g} \arrow[rr] \arrow[dr] & & \tilde{\mathbf{f}}_c, \tilde{\mathbf{g}}_c \arrow[dl] \\
& \mathbf{f} \cdot \mathbf{g} \overset{?}{=} \tilde{\mathbf{f}}_c \cdot \tilde{\mathbf{g}}_c &
\end{tikzcd}$$

commutes. Using the Fourier inversion formula we find, in fact, that

$$\begin{aligned}
\mathbf{f} \cdot \mathbf{g} &= \int_0^\infty f(x) g(x)\, dx \\
&= \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \left\{ \int_0^\infty \tilde{g}_c(y) \cos(xy)\, dy \right\} dx \\
&= \sqrt{\frac{2}{\pi}} \int_0^\infty \left\{ \int_0^\infty f(x) \tilde{g}_c(y) \cos(xy)\, dx \right\} dy,
\end{aligned}$$

provided the conditions for reversing a repeated integral are satisfied. Subject to these conditions, we may therefore conclude, using the definition of $\tilde{f}_c$, that

$$\begin{aligned}
\mathbf{f} \cdot \mathbf{g} &= \int_0^\infty \tilde{f}_c(y) \tilde{g}_c(y)\, dy \\
&= \tilde{\mathbf{f}}_c \cdot \tilde{\mathbf{g}}_c
\end{aligned}$$

so that

$$f \longmapsto \tilde{f}_c$$

is a Euclidean space morphism.

The formula we have just obtained may be written

$$\int_0^\infty f(x) g(x)\, dx = \int_0^\infty \tilde{f}_c(y) \tilde{g}_c(y)\, dy$$

or, in the special case where $g = f$

$$\int_0^\infty [f(x)]^2\, dx = \int_0^\infty [\tilde{f}_c(y)]^2\, dy \tag{1}$$

This last is called *Parseval's formula*, since it is an analogue of Parseval's equality given in *Unit 20* (page K319)

$$\|\mathbf{x}\|^2 = \sum_{k=0}^{\infty} (\mathbf{x} \cdot \mathbf{e}_k)^2.$$

We can derive Parseval's formula (1) from Parseval's equality for Fourier cosine series on $[0, p]$ as follows. From Corollary 9-1 on page K363, Parseval's equality for a Fourier cosine series takes the form

$$\int_0^p [f(x)]^2\, dx = \frac{p}{2} \left(\frac{a_0^2}{2} + a_1^2 + a_2^2 + \cdots \right) \tag{2}$$

where

$$a_k = \frac{2}{p}\int_0^p f(x)\cos\frac{k\pi x}{p}\,dx.$$

If we take the limit for large p of (2), an argument similar to that in sub-section 31.1.1 turns (2) into (1).

Exercises

1. Formulate the analogue for Fourier sine transforms of the Parseval formula for cosine transforms obtained above.
2. Show (i.e. with the same lack of rigour as in the above proof of Parseval's formula) that for real functions f, g with domain $(-\infty, \infty)$

$$\mathbf{f}\cdot\mathbf{g} = \int_{-\infty}^{\infty} \tilde{f}(y)\overline{\tilde{g}(y)}\,dy$$

where $\mathbf{f}\cdot\mathbf{g}$ now means $\int_{-\infty}^{\infty} f(x)g(x)\,dx$, and $\overline{\tilde{g}(y)}$ means the complex conjugate of $\tilde{g}(y)$.

3. We saw in the example of sub-section 31.1.3 that the Fourier transform of

$$x \longmapsto e^{-a|x|} \text{ is } y \longmapsto \sqrt{\frac{2}{\pi}}\frac{a}{a^2+y^2}.$$

Use this fact, together with the version of Parseval's formula obtained in the preceding exercise, to evaluate

$$\int_{-\infty}^{\infty}\frac{dy}{(a^2+y^2)(b^2+y^2)}.$$

Solutions

1. $$\int_0^{\infty} f(x)g(x)\,dx = \int_0^{\infty}\tilde{f}_s(y)\tilde{g}_s(y)\,dy,$$

where

$$\tilde{f}_s(y) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\sin(xy)\,dx$$

can be obtained in the same way as the formula for cosine transforms.

2. Since g is a real function, taking the complex conjugate of the result of applying the Fourier inversion formula

$$g(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\tilde{g}(y)e^{-ixy}\,dy,$$

gives

$$g(x) = \overline{g(x)} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\overline{\tilde{g}(y)}e^{ixy}\,dy$$

and so

$$\mathbf{f}\cdot\mathbf{g} = \int_{-\infty}^{\infty} f(x)\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\overline{\tilde{g}(y)}e^{ixy}\,dy\right)dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} f(x)\overline{\tilde{g}(y)}e^{ixy}\,dx\right)dy$$

(assuming that the repeated integral may be reversed)

$$= \int_{-\infty}^{\infty}\tilde{f}(y)\overline{\tilde{g}(y)}\,dy.$$

by the definition of $\tilde{f}$. (The last integral above is an example of a *Hermitian form*, as defined in Section IV-12 of N for complex vector spaces. Such forms give the generalization of Euclidean space theory to complex vector spaces, such as the one in which $\tilde{f}$ and $\tilde{g}$ lie.)

3. Writing $f(x)$ for $e^{-a|x|}$ and $g(x)$ for $e^{-b|x|}$, we have

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{dy}{(a^2+y^2)(b^2+y^2)} &= \int_{-\infty}^{\infty} \left(\sqrt{\frac{\pi}{2}}\frac{\tilde{f}(y)}{a}\right)\left(\sqrt{\frac{\pi}{2}}\frac{\overline{\tilde{g}(y)}}{b}\right) dy \\
&= \frac{\pi}{2ab}\int_{-\infty}^{\infty} \tilde{f}(y)\overline{\tilde{g}(y)}\, dy \\
&= \frac{\pi}{2ab}\int_{-\infty}^{\infty} f(x)g(x)\, dx \\
&\qquad \text{(by Parseval's formula, Exercise 2)} \\
&= \frac{\pi}{2ab}\int_{-\infty}^{\infty} e^{-(a+b)|x|}\, dx \\
&= \frac{\pi}{ab}\int_{0}^{\infty} e^{-(a+b)x}\, dx \\
&= \frac{\pi}{ab(a+b)}.
\end{aligned}$$

31.3.2 Summary of Section 31.3

In this section we defined the terms

Fourier cosine transform (re-scaled)	(page C33)
Parseval's formula	(page C34)

Technique

Use Parseval's formula in the evaluation of certain integrals.

Notation

$V[0, \infty)$ (page C33)

$E[0, \infty)$ (page C33)

Main Result

The mappings $f \longmapsto \tilde{f}_c$ and $f \longmapsto \tilde{f}_s$ with domain $E[0, \infty)$ are Euclidean space morphisms.

31.4 SUMMARY OF THE UNIT

This unit introduced another technique for solving partial differential equations: the Fourier transform method. There are two approaches to the definition of a Fourier transform. One is through investigating the Laplace transform of complex functions and the other is by investigating what happens to the convergence properties of Fourier series when the size of the interval on which they are defined is extended. It was this latter approach which we used in the unit, although we did subsequently investigate the connection between Fourier and Laplace transforms. In this way we defined the Fourier sine and cosine transforms, and these led to the definition of the Fourier transform $\bar{f}$, where

$$\bar{f}(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{ixy}\, dx.$$

The integrand in this definition is a function from the reals to the complex numbers. We therefore investigated various techniques of integration for such functions.

In the second section we illustrated the properties of the Fourier transform by using it to solve the general one-dimensional wave equation and the heat conduction equation. Since the solution to such equations involves integrals of the form

$$\int_{-\infty}^{\infty} \bar{f}(y) \times \bar{g}(y)e^{-ixy}\, dy$$

where f and g are known, we were led to define the convolution of f and g denoted by $f * g$.

$$f * g(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(z)g(x-z)\, dz = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-z)g(z)\, dz$$

The convolution theorem states:
if

$$\bar{h} = \bar{f} \times \bar{g}$$

then

$$h = f * g.$$

In the final section we set up the vector space $V[0, \infty)$ of all functions which

(i) have domain $[0, \infty)$ and codomain R,
(ii) are continuous and have piecewise continuous first and second derived functions,
(iii) are absolutely integrable on $[0, \infty)$.

In $V[0, \infty)$

$$\bar{f}_c(y) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cos(xy)\, dx \qquad (y \in R)$$

and

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \bar{f}_c(y) \cos(xy)\, dy \qquad (x \in [0, \infty))$$

The subspace $E[0, \infty)$ of $V[0, \infty)$ for which

$$\int_0^{\infty} f^2$$

converges is a Euclidean space with inner product

$$\mathbf{f} \cdot \mathbf{g} = \int_0^{\infty} fg.$$

The linear transformation

$$f \longmapsto \tilde{f}_c \qquad (f \in E[0, \infty))$$

is a Euclidean space morphism, i.e.

$$\mathbf{f} \cdot \mathbf{g} = \tilde{\mathbf{f}}_c \cdot \tilde{\mathbf{g}}_c .$$

Definitions

finitary function	(page C6)	* *
Fourier cosine transform	(pages C7 and C33)	* *
inversion formula	(page C8)	* *
Fourier sine transform	(page C8)	* *
integral transform	(page C10)	*
Fourier transform	(page C16)	* *
absolutely integrable function	(page C16)	* *
repeated integral	(page C30)	* *
convolution of two functions	(page C31)	*
Parseval's formula	(page C34)	*

Theorem

(Convolution Theorem; page C31)

If

$$\tilde{h} = \tilde{f} \times \tilde{g}$$ * * *

then

$$h = f * g$$

where

$$f * g\,(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(z) g(x - z)\, dz$$
$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x - z) g(z)\, dz.$$

Techniques

1. Integrate functions from R to C by applying the rule * *
$$\int_a^b g = \int_a^b h + ik = \int_a^b h + i \int_a^b k$$
where $g = h + ik$.
2. Apply the following techniques of integration to functions from R to C: * *
   (i) integration by substitution,
   (ii) integration by parts,
   (iii) differentiation under the integral sign.
3. Use the Fourier transform to solve partial differential equations in two variables, in particular the one-dimensional wave equation and heat conduction equations. * * *
4. Reverse repeated infinite integrals. * *
5. Use Parseval's formula in the evaluation of certain integrals. *

Notation

F_c	(page C7)
F_s	(page C8)
$\tilde{f}$	(page C16)
ϕ_t	(page C25)
ψ_y	(page C25)
$f * g$	(page C30)
$V[0, \infty)$	(page C33)
$E[0, \infty)$	(page C33)

31.5 SELF-ASSESSMENT

Self-assessment Test

This Self-assessment Test is designed to help you test your understanding of the unit. It can also be used, together with the summary of the unit, for revision. The answers to these questions will be found on the next non-facing page. We suggest that you complete the whole test before looking at the answers.

1. Find the Fourier transform of f where

$$f(x) = \begin{cases} b & \text{if } |x| \leqslant a \\ 0 & \text{if } |x| > a \end{cases} \qquad (x \in R)$$

2. (i) Assuming that the Fourier inversion formula applies to the function in Question 1 despite the fact that f is discontinuous at $\pm a$, evaluate

$$\int_{-\infty}^{\infty} \frac{\sin(ay)\cos(yx)}{y}\, dy.$$

for $|x| \neq a$.

(ii) What value would you expect for the integral in the case $x = a$?

3. Evaluate $\displaystyle\int_0^{\infty} \frac{\sin^2 ay}{y^2}\, dy.$

(*Hint:* Use Question 1 and Parseval's formula for Fourier integrals:

$$\int_{-\infty}^{\infty} [f(x)]^2\, dx = \int_{-\infty}^{\infty} [\hat{f}(y)]^2\, dy.)$$

4. Evaluate:

(i) $$\frac{d}{dy}\int_a^b \sin(xy)\, dy,$$

(ii) $$\int_a^b y e^{ixy}\, dy.$$

5. Explain the rôle that convolution could play in the evaluation of an integral in the form

$$I(x) = \int_{-\infty}^{\infty} f(y) \times g(y) e^{-ixy}\, dy$$

6. The differential equation that models transverse vibrations of an elastic rod is

$$\frac{\partial^4 u}{\partial x^4} = -\frac{1}{a^2}\frac{\partial^2 u}{\partial t^2}$$

where $u : R^2 \longrightarrow R$.

(i) Without justifying the steps in your working rigorously, use the method of Fourier transforms to obtain a general solution of this differential equation (analogous to the solution of the wave equation in Equation (7) on page C27) as the sum of two integrals each involving one arbitrary function with domain R and codomain C (the field of all complex numbers).

(ii) In the case where

$$\left.\begin{aligned} u(x,0) &= h(x) \\ \frac{\partial u}{\partial t}(x,0) &= 0 \end{aligned}\right\} \qquad (x \in R)$$

with h a known real function, deduce that a solution to the problem is

$$u(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} h(x)e^{ixy}\, dx\right]\cos(ay^2 t)e^{-ixy}\, dy.$$

Solutions to Self-assessment Test

1. The Fourier transform is $\tilde{f}$ where

$$\tilde{f}(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{ixy}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} b e^{ixy}\, dx$$

$$= \left[\frac{be^{ixy}}{(\sqrt{2\pi})(iy)}\right]_{-a}^{a} \qquad (y \neq 0)$$

$$= \frac{b}{\sqrt{2\pi}} \left(\frac{e^{iay} - e^{-iay}}{iy}\right)$$

$$= \frac{2b}{\sqrt{2\pi}} \left(\frac{\sin ay}{y}\right) \qquad (y \neq 0).$$

$$\tilde{f}(0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, dx = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} b\, dx = \frac{2ab}{\sqrt{2\pi}}.$$

2. (i) The Fourier inversion formula is

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{f}(y) e^{-ixy} dy.$$

From Question 1

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{2}{\sqrt{2\pi}} \left(\frac{\sin ay}{y}\right) e^{-ixy}\, dy = \begin{cases} 1 & |x| < a \\ 0 & |x| > a \end{cases}$$

$$\frac{1}{\pi} \int_{-\infty}^{\infty} \left(\frac{\sin ay}{y}\right) e^{-ixy}\, dy$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin (ay) \cos (yx)}{y}\, dy - \frac{i}{\pi} \int_{-\infty}^{\infty} \frac{\sin (ay) \sin (xy)}{y}\, dy.$$

The integrand in the second integral is odd and so the integral is zero.

Hence

$$\int_{-\infty}^{\infty} \frac{\sin ay \cos yx}{y}\, dy = \begin{cases} \pi & |x| < a \\ 0 & |x| > a \end{cases}$$

(ii) At the point of discontinuity $x = a$, we expect, by analogy with Theorem 9-1 on page K340, the Fourier inversion formula to converge to the mean of $f(a^+)$ and $f(a^-)$, which is $\frac{1}{2}b$.

Putting $b = 1$ we therefore expect

$$\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin ay \cos ay}{y}\, dy = \tfrac{1}{2}.$$

Hence

$$\int_{-\infty}^{\infty} \frac{\sin ay \cos ay}{y}\, dy = \frac{\pi}{2}.$$

3. From Question 1 with $b = 1$,

$$\tilde{f}(y) = \frac{2}{\sqrt{2\pi}} \left(\frac{\sin ay}{y}\right)$$

when $f(x) = \begin{cases} 1 & |x| < a \\ 0 & |x| > a \end{cases}$

Using Parseval's formula this gives

$$\int_{-a}^{a} (1)^2\, dx = \int_{-\infty}^{\infty} \frac{2}{\pi} \left(\frac{\sin^2 ay}{y^2}\right) dy.$$

But

$$\int_{-a}^{a} (1)^2\,dx = 2a$$

and

$$\int_{-\infty}^{\infty} \frac{2}{\pi}\left(\frac{\sin^2 ay}{y^2}\right) dy = \frac{4}{\pi}\int_{0}^{\infty}\left(\frac{\sin^2 ay}{y^2}\right) dy.$$

Hence

$$\int_{0}^{\infty} \frac{\sin^2 ay}{y^2}\,dy - \frac{a\pi}{2}.$$

4. (i) By differentiation under the integral sign

$$\frac{d}{dy}\int_{a}^{b} \sin(xy)\,dy = \int_{a}^{b} \frac{\partial}{\partial y}\sin(xy)\,dy$$

$$= \int_{a}^{b} x\cos(xy)\,dy$$

$$= [\sin(xy)]_a^b$$

$$= \sin(xb) - \sin(xa).$$

(ii)

$$\int_{a}^{b} ye^{ixy}\,dy = \int_{a}^{b} y\frac{d}{dy}\left(\frac{e^{ixy}}{ix}\right) dy$$

$$= \left[\frac{ye^{ixy}}{ix}\right]_a^b - \int_{a}^{b} \frac{e^{ixy}}{ix}\,dy$$

(by integration by parts)

$$= \left(\frac{1-ibx}{x^2}\right)e^{ixb} - \left(\frac{1-iax}{x^2}\right)e^{ixa}$$

5. If f and g can be recognized as the Fourier transforms of two known functions, h and k say, then

$$I(x) = \int_{-\infty}^{\infty} \tilde{h}(y) \times \tilde{k}(y)e^{-ixy}\,dy.$$

We can then apply the convolution theorem to obtain

$$I(x) = h * k(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} h(z)k(x-z)\,dz$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} h(x-z)k(z)\,dz.$$

6. (i) The transformed equation is

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{\partial^4 u}{\partial x^4}(x,t)e^{ixy}\,dy$$

$$= \frac{-1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{1}{a^2}\frac{\partial^2 u}{\partial t^2}(x,t)e^{ixy}\,dy.$$

If we introduce functions ψ_y (one for each y) and ϕ_t (one for each t) where

$$\phi_t = x \longmapsto u(x,t) \qquad (x \in R)$$

and

$$\psi_y : y \longmapsto \tilde{\phi}_t(y) \qquad (y \in R)$$

the transformed equation becomes (using integration by parts four times on the left-hand side and differentiation under the integral on the right-hand side)

$$(-iy)^4\psi_y(t) = -\frac{1}{a^2}\psi_y''(t)$$

i.e.

$$\psi_y''(t) = -(ay^2)^2\psi_y(t).$$

This ordinary differential equation has general solution (one for each y)

$$\psi_y(t) = G_y e^{iay^2t} + F_y e^{-iay^2t}.$$

We generalize the arbitrary constants G_y and F_y by writing

$$\psi_y(t) = \tilde{g}(y)e^{iay^2t} + \tilde{f}(y)e^{-iay^2t}$$

and then use

$$\psi_y(t) = \tilde{\phi}_t(y)$$

and

$$u(x, t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ixy}\tilde{\phi}_t(y)\,dy$$

to obtain

$$u(x, t) = \frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{\infty} e^{-i(x-ayt)y}\tilde{g}(y)\,dy + \int_{-\infty}^{\infty} e^{-i(x+ayt)y}\tilde{f}(y)\,dy\right]$$

(ii) Substituting for $u(x, 0)$ and $\frac{\partial u}{\partial t}(x,0)$ (using integration under the integral sign in the second case) we find

$$h(x) = \frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{\infty} \tilde{g}(y)e^{-ixy}\,dy + \int_{-\infty}^{\infty} \tilde{f}(y)e^{-ixy}\,dy\right]$$

$$0 = \frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{\infty} \tilde{g}(y)iay^2e^{-ixy}\,dy + \int_{-\infty}^{\infty} \tilde{f}(y)(-iay^2)e^{-ixy}\,dy\right]$$

If f and g are the inverse Fourier transforms of $\tilde{f}$ and $\tilde{g}$, then the first formula gives

$$h(x) = g(x) + f(x)$$

and the second can be satisfied by making

$$\tilde{f}(y) = \tilde{g}(y)$$

i.e.

$$f(x) = g(x).$$

Hence

$$f = g = \tfrac{1}{2}h.$$

Substituting this in the solution obtained in (i) and using Euler's formula

$$\tfrac{1}{2}(e^{i\theta} + e^{-i\theta}) = \cos\theta$$

with $\theta = ay^2t$, we obtain the given formula.

31.6 APPENDICES (Optional)

Appendix 1: proof of formula (7) in sub-section 31.1.1.

(We expect this to be of interest primarily to those students who enjoy analysis.)

We wish to prove that, for any $x \geqslant 0$,

$$\int_a^\infty g = \lim_{h \to 0} h\{g(a) + g(a+h) + g(a+2h) + \cdots\} \tag{1}$$

where

$$g(y) = \cos(xy) \int_0^\infty f(z) \cos(zy)\, dz \tag{2}$$

We have assumed that f is continuous, piecewise twice continuously differentiable, and finitary. Hence in $[0, \infty)$ f'' is continuous except at a finite set of points, which we call, in increasing order,

$$x_1, x_2, \ldots, x_{n-1}$$

where

$$x_0 = 0,\ x_n = p_0$$

with $[-p_0, p_0]$ the interval outside which $f(x) = 0$.

We shall prove three lemmas

(i) g is continuous (this ensures that $\int_a^b g$ exists).

(ii) There is a number A such that

$$|g(y)| < \frac{A}{y^2}$$

for all y (this ensures that $\lim_{b \to \infty} \int_a^b g$ exists).

(iii) The "tail" of the series, i.e.

$$T_n = h\{g(a+nh) + g(a+(n+1)h) + \cdots\}$$

is small for large n (this ensures convergence of the series on the right of (1)).

Proof of (i)

g is the product of the continuous function

$$y \longmapsto \cos(xy)$$

and the function

$$F_c : g \longmapsto \int_0^{p_0} f(x) \cos(xy)\, dx,$$

so it is sufficient to prove that F_c is continuous. This is done as follows: for any ε we have

$$\begin{aligned} F_c(y+\varepsilon) - F_c(y) &= \int_0^{p_0} f(x)[\cos x(y+\varepsilon) - \cos(xy)]\, dx \\ &= -2\int_0^{p_0} f(x) \sin x(y + \tfrac{1}{2}\varepsilon) \sin(\tfrac{1}{2}x\varepsilon)\, dx \end{aligned}$$

since

$$2 \sin r \sin s = \cos(r-s) - \cos(r+s)$$

for all real r, s.

Hence

$$|F_c(y+\varepsilon) - F_c(y)| \leqslant 2\int_0^{p_0} |f(x)|\,|\sin x(y+\tfrac{1}{2}\varepsilon)|\,|\sin(\tfrac{1}{2}x\varepsilon)|\,dx$$

$$\leqslant 2\int_0^{p_0} |f(x)|\,\tfrac{1}{2}x\varepsilon\,dx \quad \text{since} \quad |\sin\theta| \leqslant |\theta|$$

$$\leqslant p_0\,\varepsilon \int_0^{p_0} |f(x)|\,dx$$

and hence

$$\lim_{\varepsilon\to 0} F_c(y+\varepsilon) = F_c(y)$$

for all y, which is the definition of continuity.

Proof of (ii)

Integration by parts gives

$$F_c(y) = \int_0^{p_0} f(x)\cos(xy)\,dx$$

$$= \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} f(x)\cos(xy)\,dx$$

$$= \sum_{i=1}^{n} \left[f(x)\frac{\sin(xy)}{y} - \int f'(x)\frac{\sin(xy)}{y}\,dx \right]_{x_{i-1}}^{x_i}$$

$$= -\frac{1}{y}\sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} f'(x)\sin(xy)\,dx$$

since $\sin 0 = 0$ and $f(x_n) = 0$ and the other terms not involving integrals cancel in pairs because of the continuity of f.

A second integration by parts gives

$$F_c(y) = -\frac{1}{y^2}\sum_{i=1}^{n} \left[-f'(x)\cos(xy) + \int f''(x)\cos(xy)\,dx \right]_{x_{i-1}}^{x_i}$$

This time there is no cancellation since f' need not be continuous at $x_1, x_2, \ldots, x_{n-1}$, but the magnitude of the ith square bracket is bounded above by

$$|f'(x_i^-)| + |f'(x_{i-1}^+)| + \int_{x_{i-1}}^{x_i} |f''(x)|\,dx$$

and therefore the whole sum is bounded in magnitude by the sum of these expressions, which we may call A. We conclude, using (2), that

$$|g(y)| \leqslant |F_c(y)| \leqslant \frac{A}{y^2}.$$

Proof of (iii)

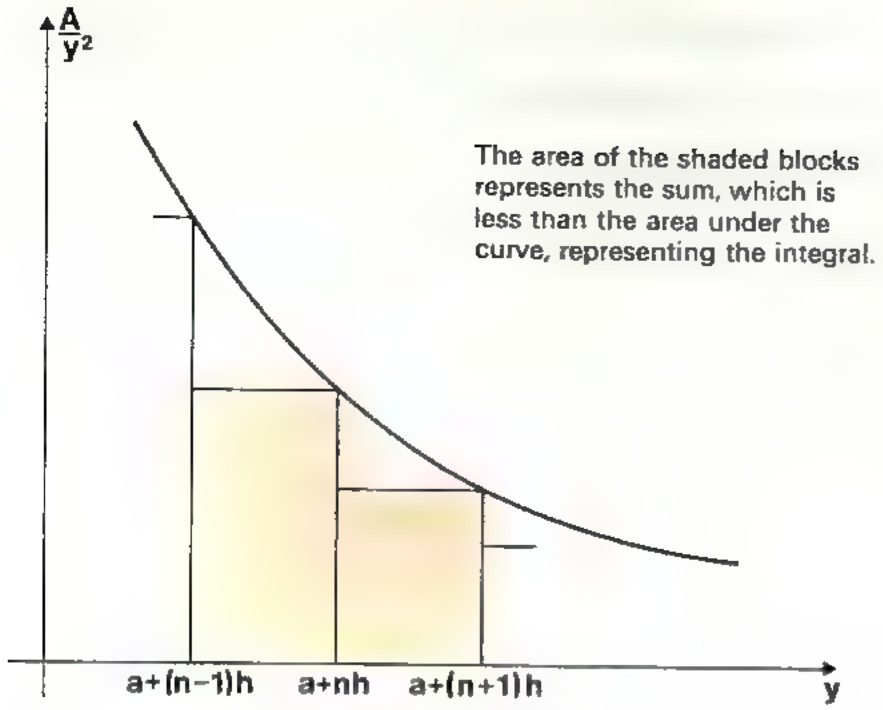
$$|T_n| \leqslant h(|g(a+nh)| + |g(a+(n+1)h)| + \cdots)$$

$$\leqslant h\left(\frac{A}{(a+nh)^2} + \frac{A}{(a+(n+1)h)^2}\right) \qquad \text{by (ii)}$$

$$\leqslant \int_{a+(n-1)}^{\infty} h\frac{A}{y^2}\,dy \qquad \text{(see diagram)}$$

$$= \frac{A}{a+(n-1)h}$$

where we have assumed that n is large enough to make $a + (n-1)h > 0$.



Using these three lemmas we can now prove (1) itself. We shall show that, for large b and n

$$\int_a^\infty g \approx \int_a^b g \approx h(g(a) + g(a+h) + \cdots + g(a+(n-1)h))$$

$$\approx h(g(a) + g(a+h) + \cdots) \qquad (3)$$

where $h = (b-a)/n$ and the symbol $\approx$ means that the expressions it connects can be made as close together as we please by choosing b and n large enough.

From Lemma (ii) we have

$$\int_a^\infty g - \int_a^b g = \int_b^\infty g \quad \text{and} \quad \left|\int_b^\infty g\right| \leqslant \int_b^\infty \frac{A}{y^2}\,dy = \frac{A}{b}.$$

Also, by lemma (iii) we know that T_n, the difference between the finite and the infinite series in (3), satisfies

$$|T_n| \leqslant \frac{A}{a+(n-1)h} = \frac{A}{b-h} \leqslant \frac{A}{b-1} \text{ if } h < 1.$$

Thus, by choosing b large enough, we can make both of these differences as small as we please provided that h is less than 1. The remaining difference to consider is

$$\int_a^b g - h(g(a) + \cdots + g(a+(n-1)h))$$

and, once b has been chosen, we can make this difference as small as we please by making n large enough. (This is just the definition of $\int_a^b g$, given in *Unit M100 9, Integration I.*) Thus by suitably choosing first b and then n we can make all the members of (3) as close together as we please. Since the first and last members do not depend on b or n, this is only possible if these two members are equal.

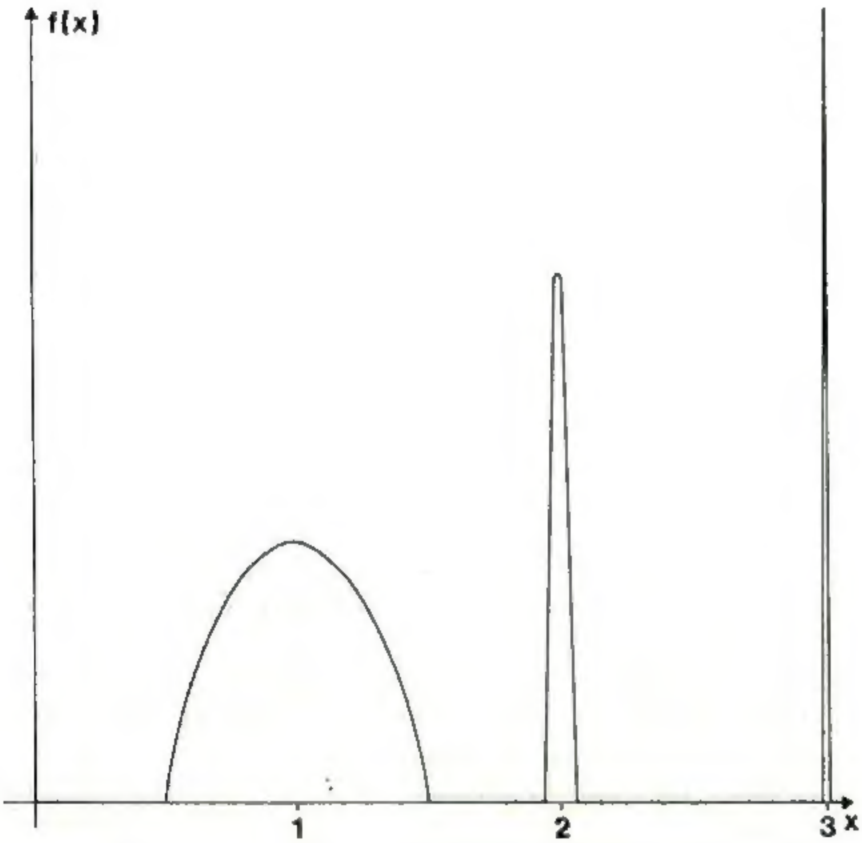
Appendix 2

A function that is absolutely integrable, but not square-integrable, on $[0, \infty)$ is

$$f = \sum_{n=1}^{\infty} f_n$$

where

$$f_n : x \longmapsto \begin{cases} 0 \text{ if } 0 \leqslant x < n - \dfrac{1}{2n^3} \\ 0 \text{ if } n + \dfrac{1}{2n^3} < x < \infty \\ n(1 - 4n^6(x-n)^2) \text{ if } n - \dfrac{1}{2n^3} \leqslant x \leqslant n + \dfrac{1}{2n^3} \end{cases} \quad (x \in [0, \infty))$$



The graph of f consists of a succession of parabolic arches, the nth of which has height n and width $\frac{1}{n^3}$, so that its area is proportional to

$$n \times \frac{1}{n^3} = \frac{1}{n^2} \quad \left(\text{actually } \frac{2}{3n^2}\right).$$

Thus we have

$$\int_0^\infty |f| = \int_0^\infty f = \sum_{n=1}^{\infty} \frac{2}{3n^2} = \frac{\pi^2}{9}.$$

The graph of f^2, on the other hand, consists of a succession of arches of height n^2 and width $\frac{1}{n^3}$; the integral

$$\int_{\frac{1}{2}}^{N+\frac{1}{2}} f^2$$

is therefore proportional to

$$\sum_{n=1}^{N} n^2 \frac{1}{n^3} = \sum_{n=1}^{N} \frac{1}{n},$$

and since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges the integral $\int_{\frac{1}{2}}^{\infty} f^2$ diverges, and hence $\int_{0}^{\infty} f^2$ diverges too.

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